SOURCE TERMS AND MULTIPLICITY OF SOLUTIONS
IN A NONLINEAR ELLIPTIC EQUATION

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\textbf{ABSTRACT:} We are concerned with the multiplicity of solutions of a nonlinear elliptic equation. We investigate relations between the multiplicity of solutions and source terms in the Dirichlet problem.

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1. INTRODUCTION

Let $\Omega$ be a bounded set in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\partial \Omega$ and let $A$ denote the elliptic operator

\[ A = \sum_{1 \leq i,j \leq n} a_{ij}(x) D_i D_j, \quad (1.1) \]

where $a_{ij} = a_{ji} \in C^\infty(\tilde{\Omega})$.

We consider a semilinear elliptic boundary value problem under the Dirichlet boundary condition

\[ Au + bu^+ - au^- = h(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega. \quad (1.2) \]

Here $A$ is a second order elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated as often as multiplicity. We denote $\phi_n$ to be the eigenfunction corresponding to $\lambda_n (n = 1, 2, \cdots)$, and $\phi_1$ is the eigenfunction such that $\phi_1 > 0$ in $\Omega$ and the set $\{ \phi_n \mid n = 1, 2, 3 \cdots \}$ is an orthonormal set in $H$, where $H$ is a Hilbert space with inner product

\[ (u, v) = \int_{\Omega} uv, \quad u, v \in L^2(\Omega). \]
We suppose that $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Under these assumptions, we have a concern with the multiplicity of solutions of (1.2) when $h$ is generated by two eigenfunctions $\phi_1$ and $\phi_2$. Then equation (1.2) is equivalent to

$$Au + bu^+ - au^- = h \quad \text{in} \quad H,$$

(1.3)

where $h = t_1\phi_1 + t_2\phi_2(t_1, t_2 \in \mathbb{R})$. Hence we will study the equation (1.3). To study equation (1.3), we use the contraction mapping principle to reduce the problem from an infinite dimensional space in $H$ to a finite dimensional one.

Let $V$ be the two dimensional subspace of $H$ spanned by $\{\phi_1, \phi_2\}$ and $W$ be the orthogonal complement of $V$ in $H$. Let $P$ be an orthogonal projection $H$ onto $V$. Then every element $u \in H$ is expressed as

$$u = v + w,$$

where $v = Pu, w = (I - P)u$. Hence equation (1.3) is equivalent to a system

$$Aw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0$$

(1.4)

$$Av + P(b(v + w)^+ - a(v + w)^-) = t_1\phi_1 + t_2\phi_2.$$  

(1.5)

Here we look on (1.4) and (1.5) as a system of two equation in the two unknows $v$ and $w$.

For fixed $v \in V$, (1.4) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the $L^2$-norm) in terms of $v$.

The study of the multiplicity of solution of (1.3) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = t_1\phi_1 + t_2\phi_2$$

(1.6)

defined on the two dimensional subspace $V$ spanned by $\{\phi_1, \phi_2\}$.

While one feels intuitively that (1.6) ought to be easier to solve than (1.3), there is the disadvantage of an implicitly defined term $\theta(v)$ in the equation. However, in our case, it turns out that we know $\theta(v)$ for some special $v$’s.

If $v \geq 0$ or $v \leq 0$, then $\theta(v) \equiv 0$. For example, let us take $v \geq 0$ and $\theta(v) = 0$. Then equation (1.4) reduces to

$$A0 + (I - P)(bv^+ - av^-) = 0,$$

which is satisfied because $v^+ = v, v^- = 0$ and $(I - P)v = 0$, since $v \in V$. Since the subspace $V$ is spanned by $\{\phi_1, \phi_2\}$ and $\phi_1$ is a positive eigenfunction, there exists a cone $C_1$ defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \geq 0, |c_2| \leq qc_1\}$$

for some $q > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone $C_3$ defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 \mid c_1 \leq 0, |c_2| \leq q|c_1|\},$$
so that $v \leq 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$. Now we define a map $\Pi : V \to V$ given by

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-)$$

$v \in V$.  \hspace{1cm} (1.7)

2. THE NONLINEARITY CROSSES ONE EIGENVALUE

**Theorem 2.1.** $\Pi(cv) = c\Pi(v)$ for $c \geq 0$.

**Proof.** Let $c \geq 0$. If $v$ satisfies

$$A(\theta(v)) + (I - P)(b(v + \theta(v))^+ - a(v + \theta(v))^-) = 0,$$

then

$$A(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\Pi(cv) = A(cv) + P(b(cv + \theta(cv))^+ - a(cv + \theta(cv))^-)$$

$$= cAv + P(b(cv + c\theta(v))^+ - a(cv + c\theta(v))^-)$$

$$= c\Pi(v).$$

We investigate the image of the cones $C_1, C_3$ under $\Pi$. First, we consider the image of cone $C_1$. If $v = c_1\phi_1 + c_2\phi_2 \geq 0$, we have

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-)$$

$$= -c_1\lambda_1\phi_1 - c_2\lambda_2\phi_2 + b(c_1\phi_1 + c_2\phi_2)$$

$$= c_1(b - \lambda_1)\phi_1 + c_2(b - \lambda_2)\phi_2.$$

Thus the image of the rays $c_1\phi_1 = q_1c_1\phi_2(c_1 \geq 0)$ can explicitly caculated and they are

$$c_1(b - \lambda_1)\phi_1 = q_1c_1(b - \lambda_2)\phi_2 (c_1 \geq 0).$$

(2.1)

Therefore If $\lambda_1 < a < \lambda_2 < b < \lambda_3$, then $\Pi$ maps $C_1$ onto the cone

$$R_1 = \left\{ d_1\phi_1 + d_2\phi_2 \mid d_1 \geq 0, |d_2| \leq q \left( \frac{b - \lambda_2}{b - \lambda_1} \right) d_1 \right\}.$$

Second, we consider the image of the cone $C_3$. If

$$v = -c_1\phi_1 + c_2\phi_2 \leq 0 \hspace{1cm} (c_1 \geq 0, |c_2| \leq q_{c_1}),$$

the image of the rays $-c_1\phi_1 = q_1c_1\phi_2(c_1 \geq 0)$ are

$$c_1(\lambda_1 - a)\phi_1 = q_1c_1(\lambda_2 - a)\phi_2 \hspace{1cm} (c_1 \geq 0).$$

(2.2)

Therefore, if $\lambda_1 < a < \lambda_2 < b < \lambda_3$, then $\Pi$ maps the cone $C_3$ onto the cone

$$R_3 = \left\{ d_1\phi_1 + d_2\phi_2 \mid d_1 \leq 0, |d_2| \leq q \left( \frac{\lambda_2 - a}{\lambda_1 - a} \right) d_1 \right\}.$$
Now we set

$$C_2 = \{ v = c_1 \phi_1 + c_2 \phi_2 \mid c_2 \geq 0, c_2 \geq q|c_1| \},$$

$$C_4 = \{ v = c_1 \phi_1 + c_2 \phi_2 \mid c_2 \leq 0, |c_2| \geq q|c_1| \},$$

Then the union of $C_1, C_2$, and $C_3, C_4$ are the space $V$.

We remember the map $\Pi : V \rightarrow V$ given by

$$\Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) \quad v \in V.$$

Let $R_i \ (1 \leq i \leq 4)$ be the image of $C_i (1 \leq i \leq 4)$ under $\Pi$.

**Theorem 2.2.** Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. If $h$ belongs to $R_1$, then equation (1.2) has a positive solution and no negative solution. If $h$ belongs to $R_3$, then equation (1.2) has a negative solution.

**Proof.** From (2.1) and (2.2), if $h$ belongs to $R_1$, the equation $\Pi(v) = t_1 \phi_1 + t_2 \phi_2$ has a positive solution in the cone $C_1$, namely $\frac{t_1}{b-a} \phi_1 + \frac{t_2}{b-a} \phi_2$, and if $h$ belongs to $R_3$, the equation $\Pi(v) = t_1 \phi_1 + t_2 \phi_2$ has a negative solution in $C_3$, namely $-\frac{t_1}{b-a} \phi_1 - \frac{t_2}{b-a} \phi_2$.

Lemma 2.1 means that the images $\Pi(C_2)$ and $\Pi(C_4)$ are the cones in the plane $V$.

Before we investigate the images $\Pi(C_2)$ and $\Pi(C_4)$, we set

$$R_2^* = \left\{ d_1 \phi_1 + d_2 \phi_2 \mid d_2 \geq 0, -q^{-1} \mid \frac{\lambda_1}{\lambda_2-a} \mid d_2 \leq d_1 \leq q^{-1} \mid \frac{b-\lambda_2}{b-a} \mid |d_2| \right\},$$

$$R_4^* = \left\{ d_1 \phi_1 + d_2 \phi_2 \mid d_2 \leq 0, -q^{-1} \mid \frac{\lambda_1}{\lambda_2-a} \mid |d_2| \leq d_1 \leq q^{-1} \mid \frac{b-\lambda_1}{b-a} \mid |d_2| \right\}.$$

Then the union of $R_1, R_2^*, R_3, R_4^*$ is the plane $V$.

To investigate a relation between the multiplicity of solutions and source terms in a nonlinear elliptic differential equation

$$Au + bu^+ - au^- = h \quad \text{in} \quad H,$$

we consider the restriction $\Pi|_{C_i} (1 \leq i \leq 4)$ of $\Pi$ to the cone $C_i$. Let $\Pi_i = \Pi|_{C_i}$, i.e.,

$$\Pi_i : C_i \rightarrow V.$$

**Theorem 2.3.** For $i = 1, 3$, the image of $\Pi_i$ is $R_i$ and $\Pi_i : C_i \rightarrow R_i$ is bijective.

**Proof.** We consider the restriction $\Pi_1$. By (2.4), the restriction $\Pi_1$ maps $C_1$ onto $R_1$.

Let $l_1$ be the segment defined by

$$l_1 = \left\{ \phi_1 + d_2 \phi_2 \mid |d_2| \leq q \left( \frac{b-\lambda_2}{b-a} \right) \right\}.$$

Then the inverse image $\Pi_1^{-1}(l_1)$ is a segment

$$L_1 = \left\{ \frac{1}{b-\lambda_1} (\phi_1 + c_2 \phi_2) \mid |c_2| \leq q \right\}.$$

It follows from Theorem 2.1 that $\Pi_1 : C_1 \rightarrow R_1$ is bijective.

Similarly, $\Pi_3 : C_3 \rightarrow R_3$ is also a bijection. \qed
We have investigated next lemma in [5].

**Lemma 2.4.** Let $Q_2$ be one of the sets $R_1 \cup R_4^*$ or $R_2^* \cup R_3$ such that it is contained in $\Pi(C_2)$ and let $Q_4$ be one of the sets $R_1 \cup R_2^*$ or $R_3 \cup R_4^*$ such that it is contained in $\Pi(C_4)$. Let $\gamma_i (i = 2, 4)$ be any simple path in $Q_i$ with end points on $\partial Q_i$, where each ray (starting from the origin) in $Q_i$ intersects only one point of $\gamma_i$. Then the inverse image $\Pi_i^{-1}(\gamma_i)$ of $\gamma_i$ is a simple path in $C_i$ with end points on $\partial C_i$, where any ray (starting from the origin) in $C_i$ intersects only one point of this path.

By Lemma 2.4, we have the following theorem.

**Theorem 2.5.** For $i = 2, 4$, if we let $\Pi_i(C_i) = R_i$, then $R_2$ is one of the sets $R_1 \cup R_4^*$ or $R_2^* \cup R_3$, and $R_4$ is one of the sets $R_3 \cup R_4^*$ or $R_1 \cup R_2^*$. Furthermore the restriction $\Pi_i$ maps $C_i$ onto $R_i$.

### 3. Solutions and Applications of Critical Points Theory

We investigate the multiplicity of solutions of a nonlinear elliptic differential equation

$$Au + bu^+ - au^- = t\phi_1 \quad \text{in} \quad H,$$

where $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t > 0$.

Henceforth, let $F$ denote the functional defined by

$$F(u) = \int_{\Omega} \left[ \frac{1}{2} |\nabla u|^2 - G(u) + t\phi_1 u \right] \, dx,$$

where $G(u) = \frac{1}{2} (b(u^+)^2 + a(u^-)^2)$ and $u \in E$. Then,

$$DF(u)y = F'(u)y = \int_{\Omega} (\nabla u \cdot \nabla y - g(u)y + t\phi_1 y) \, dx$$

for all $y \in E$ and solutions of (3.1) coincide with solutions of

$$DF(u) = 0,$$

where $g(u) = G'(u) = bu^+ - au^-$. Therefore, we shall investigate critical points of $F$.

**Theorem 3.1.** Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$, $h \in V$. Let $v \in V$ be given. Then there exists a unique solution $z \in W$ of the equation

$$Az + (I - P)(b(v + z)^+ - a(v + z)^- - h) = 0 \quad \text{in} \quad W.$$  

If $z = \theta(v)$, then $\theta$ is continuous on $V$ and we have $DF(v + \theta(v))(w) = 0$ for all $w \in W$. In particular $\theta(v)$ satisfies a uniform Lipschitz in $v$ with respect to the $L^2$-norm. If $\widetilde{F} : V \rightarrow R$ is defined by $\widetilde{F}(v) = F(v + \theta(v))$, then $\widetilde{F}$ has continuous Frechét derivative $D\widetilde{F}$ with respect to $v$ and

$$D\widetilde{F}(v)(r) = DF(v + \theta(v))(r) \quad \text{for all} \quad r \in V.$$
If $v_0$ is a critical point of $\tilde{F}$, then $v_0 + \theta(v_0)$ is a solution of (3.1) and conversely every solution of (3.1) is $D\tilde{F}(v_0) = 0$.

**Proof.** Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$, $\alpha = \frac{1}{2}(a + b)$, and $g(u) = bu^+ - au^-$. If $g_1(u) = g(u) - \alpha u$, then equation (3.4) is equivalent to

$$z = (-A - \alpha)^{-1}(I - P)(g_1(v + w)).$$

The right hand side of (3.5) defines, for fixed $v \in V$, a Lipschitz mapping of $(I - P)H$ into itself with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I - P)H$ which satisfies (3.5). If $\theta(v)$ denotes the unique $\in (I - P)H$ which solves (3.5) then $\theta$ is continuous (with respect to the $L^2$-norm) in $V$. In fact, $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$, then we have

$$z_1 - z_2 = (-A - \alpha)^{-1}(I - P)[(g_1(v_1 + z_1) - g_2(v_2 + z_2)]$$

$$= (-A - \alpha)^{-1}(I - P)[(g_1(v_1 + z_1) - g_1(v_1 + z_2)]$$

$$+ (-A - \alpha)^{-1}(I - P)[(g_1(v_1 + z_2) - g_1(v_2 + z_2)].$$

Since $|g_1(u_1) - g_1(u_2)| \leq (b - \alpha)|u_1 - u_2|$, it follows that if $\beta = \max\{(|\lambda_m - \alpha|)\} m \geq 3, m \in N = (\lambda_3 - \alpha)^{-1} = ||(-A - \delta)^{-1}(I - P)||$, and $\gamma = \beta(b - \alpha) < 1$, then

$$||z_1 - z_2|| \leq \gamma (||v_1 - v_2|| + ||z_1 - z_2||).$$

Hence

$$||z_1 - z_2|| \leq k||v_1 - v_2||, \quad k = \frac{\gamma}{1 - \gamma},$$

which shows that $\theta(v)$ satisfies a uniform Lipschitz condition in $v$ with respect to the $L^2$ norm. Since $\theta$ is continuous on $V$, $\tilde{F}$ is $C^1$ with respect to $v$ and

$$D\tilde{F}(v)(r) = DF(v + \theta(v))(r) \text{ for all } r \in V. \quad (3.6)$$

Suppose that there exists $v_0 \in V$ such that $D\tilde{F}(v_0) = 0$. From (3.3) and (3.6) it follows that $D\tilde{F}(v_0)(v) = DF(v_0 + \theta(v_0))(v) = 0$ for all $v \in V$. Since

$$\int_\Omega \nabla v \cdot \nabla w = 0 \text{ for all } w \in W,$$

we have

$$DF(v + \theta(v))(w) = 0 \text{ for all } w \in W.$$  

Since $H$ is direct sum of $V$ and $W$, it follows that $DF(v_0 + \theta(v_0)) = 0$ in $H$. Therefore, $u = v_0 + \theta(v_0)$ is a solution of (3.1).

Conversely our reasoning shows that if $u$ is a solution of (3.1) and $v = Pu$, then $D\tilde{F}(v) = 0$ in $V$. \qed

Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $h$ belongs to the cone $R_1$. Then equation (3.1) has a positive solution $u_p$ in the cone $C_1$. By Theorem 3.1, $u_p$ can be written by $u_p = v_p + \theta(v_p).$ Since $v_p \in C_1, \theta(v_p) = 0.$ Therefore we have $u_p = v_p.$ Similary, if $h \in R_3$, then (3.1) has a negative solution $u_n$ and $u_n = v_n + \theta(v_n),$ where $\theta(v_n) = 0.$
Theorem 3.2. Let \( \lambda_1 < a < \lambda_2 < b < \lambda_3 \). Then we have:

(a) Let \( t = b - \lambda_1(h = (b - \lambda_1)\phi_1) \). Then equation (3.1) has a positive solution \( v_p \) and there exists a small open neighborhood \( B_p \) of \( v_p \) in \( C_1 \) such that in \( B_p, v_p \) is a strict local point of maximum of \( \tilde{F} \).

(b) \( t = \lambda_1 - a(h = (\lambda_1 - a)\phi_1) \). Then equation (3.1) has a negative solution \( v_n \) and there exists a small open neighborhood \( B_n \) of \( v_n \) in \( C_3 \) such that in \( B_n, v_n \) is a saddle point of \( \tilde{F} \).

Proof. (a) Let \( t = b - \lambda_1(h = (b - \lambda_1)\phi_1) \). Then equation (3.1) has a solution \( u_p = \phi_1 \) which is of the form \( u_p = v_p + \theta(v_p) \) (in this case \( \theta(v_p) = 0 \)) and \( I + \theta \), where \( I \) is an identity map on \( V \), is continuous. Since \( v_p \) is in the interior of \( C_1 \), there exists a small open neighborhood \( B_p \) of \( v_p \) in \( C_1 \). We note that \( \theta(v) = 0 \) in \( B_p \). Therefore, if \( v = v_p + v^* \in B_p \), then we have

\[
\tilde{F}(v) = \tilde{F}(v_p + v^*)
\]

\[
= \int_{\Omega} \left[ \frac{1}{2}(|\nabla(v_p + v^*)|^2 - b((v_p + v^*)^2 - a((v_p + v^*)^2) + h(v_p + v^*) \right] dx
\]

\[
= \frac{1}{2} \int_{\Omega} (||v^*|^2 - bv^*_2)dx + \int_{\Omega} [\nabla v_p \cdot \nabla v^* - bv^*_2 + hv^*] dx
\]

\[
= \frac{1}{2} \int_{\Omega} (||v^*|^2 - bv^*_2)dx + \int_{\Omega} [\nabla v_p \cdot \nabla v^* - bv^*_2 + hv^*] dx + C,
\]

where \( C = \int_{\Omega} \left[ \frac{1}{2}(|\nabla v_p|^2 - bv^*_2) + hv_p \right] dx = F(u_p) = \tilde{F}(v_p) \).

If \( v \in V \) and \( v = c_1\phi_1 + c_2\phi_2 \), then we have

\[
||v||^2_0 = \int_{\Omega} |\nabla v|^2 dx = \sum_{i=1}^{2} c_i^2 \lambda_i < \lambda_2 \sum_{i=1}^{2} c_i^2
\]

\[
= \lambda_2 \int_{\Omega} v^2 dx = \lambda_2 ||v||^2.
\]

(3.7)

Let \( v^* = c_1\phi_1 + c_2\phi_2 \) and let \( v = v_p + v^* \in B_p \). Then

\[
\int_{\Omega} [\nabla v_p \cdot \nabla v^* - bv^*_2 + hv^*] dx = 0.
\]

By (3.7),

\[
\tilde{F}(v) - \tilde{F}(v_p) = \frac{1}{2} \int_{\Omega} (||v^*_2|^2 - bv^*_2)dx < (\lambda_2 - b) \int_{\Omega} v^2 dx.
\]

Since \( \lambda_2 < b \), it follows that for \( t = b - \lambda_1 \), \( v_p \) is a strict local point of maximum for \( \tilde{F}(v) \).

(b) Let \( t = \lambda_1 - a(h = (\lambda_1 - a)\phi_1) \). Then equation (3.1) has a negative solution \( u_n = -\phi_1 \) which is of the form \( u_n = v_n + \theta(v_n) \), where \( \theta(v_n) \) and \( -I + \theta \) is continuous.
in $V$. Since $v_n$ is the interior, $\text{Int}C_3$, of $C_3$. We note that $\theta(v) = 0$ in $B_n$. Therefore, if $v = v_n + v_* \in B_n$, then we have

$$
\bar{F}(v) = \bar{F}(v_n + v_*)
$$

$$
= \int_\Omega \left[ \frac{1}{2}(|\nabla (v_n + v_*)|^2 - a((v_n + v_*)^-)^2) + h(v_n + v_*) \right] \, dx
$$

$$
= \frac{1}{2} \int_\Omega (|\nabla v_*|^2 - av_*^2) \, dx + \int_\Omega [\nabla v_n \cdot \nabla v_* - av_nv_* + hv_*] \, dx + \bar{F}(v_n).
$$

Let $v_* = c_1 \phi_1 + c_2 \phi_2$. Then for $v = v_n + v_*$, we have

$$
\int_\Omega [\nabla v_n \cdot \nabla v_* - av_nv_* + hv_*] \, dx = 0.
$$

Therefore,

$$
\bar{F}(v) - \bar{F}(v_n) = \frac{1}{2} \int_\Omega (|\nabla v_*|^2 - av_*^2) \, dx
$$

$$
= \frac{1}{2} (c_1^2(\lambda_1 - a) + c_2^2(\lambda_2 - a)).
$$

The above equation implies that $v_n$ is a saddle point of $\bar{F}$.

**Theorem 3.3.** Let $h \in V$ and let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. For fixed $t$ the functional $\bar{F}$, defined on $V$, satisfies the Palais-Smale condition: Any sequence $\{v_n\}_1^\infty \subset V$ for which $\bar{F}(v_n)$ is bounded and $\bar{F}'(v_n) \to 0$ possesses a convergent subsequence.

**Proof.** It is enough to show that if $\{v_n\}_1^\infty$ is a sequence in $V$ such that $\{D\bar{F}(v_n)\}_1^\infty$ is bounded, then the sequence of norms $\{|v_n|_0\}_1^\infty$ is bounded. Assuming the contrary, we may suppose that $\{D\bar{F}(v_n)\}_1^\infty$ is bounded and $|v_n|_0 \to \infty$ as $n \to \infty$. Since all norms on the finite dimensional space $V$ equivalent it follows that $|v_n| \to \infty$ as $n \to \infty$, where $|\cdot|$ is $L^2(\Omega)$ norm. If for each $n \geq 1$ we set $z_n = \theta(v_n)$ and $u_n = v_n + \theta(v_n)$, then $|u_n| \to \infty$ as $n \to \infty$. Therefore, since $|v_n|/|u_n|^2 \to 0$ as $n \to \infty$, $\bar{F}(v_n)(v)/|v_n|^2 \to 0$ as $n \to \infty$. Since $\bar{F}(v_n)(v) = F(u_n)(v)$ for all $v \in V$, so setting $w_n = u_n/|u_n|$. We conclude that

$$
\int_\Omega \left[ (\nabla w_n \cdot \nabla v_n - bw_n^+v_n + aw_n^-v_n + t\phi_1(v_n/|u_n|))/|u_n|| \right] \, dx \to 0 \quad (3.8)
$$

as $n \to \infty$.

We see that

$$
\int_\Omega (\nabla u_n \cdot \nabla z_n - bu_n^+z_n + au_n^-z_n + t\phi_1z) \, dx = 0 \quad \text{for all } n. \quad (3.9)
$$

Dividing the left-hand side (3.9) by $|u_n|^2$, adding to the left-hand side of (3.8) and using $w_n = v_n/|u_n| + z_n/|u_n|$, we see that (3.8) can be rewritten in the form

$$
\int_\Omega \left[ |\nabla w_n|^2 - b(u_n^+)^2 - a(w_n^-)^2 + t\phi_1w_n/|u_n|| \right] \, dx \to 0 \quad \text{as } n \to \infty.
$$
Since \(||w_n|| = 1\) for all this implies that

\[ ||w_n||^2 = \int_\Omega |\nabla w_n|^2 dx \]

is bounded independently of \(n\). Therefore, we may assume, without loss of generality, that \(\{w_n\}_1^\infty\) converges weakly to \(w \in W\). Since the injection from \(H\) into \(L^2(\Omega)\) is compact, it follows that \(\{w_n\}_1^\infty\) converges strongly in \(L^2(\Omega)\) and \(||w|| = 1\). If \(z \in W\), then, by the proof of Theorem 3.1,

\[ \int_\Omega (\nabla u_n \cdot \nabla z - b u_n^+ z + a u_n^- z + t \phi_1 z) dx = 0. \]

Dividing by \(||u_n||\) we have

\[ \int_\Omega (\nabla w_n \cdot \nabla z - b u_n^+ z + a w_n^- z + t \phi_1 z/||u_n||) dx = 0 \]  \hspace{1cm} (3.10)

for all \(n\). Letting \(n \to \infty\) in the last equation, we conclude that

\[ \int_\Omega (\nabla w \cdot \nabla z - b w^+ z + a w^- z) dx = 0. \]  \hspace{1cm} (3.11)

Let \(v \in V\). We see that

\[ D\tilde{F}(v_n)(v) = \int_\Omega (\nabla u_n \cdot \nabla v - b u_n^+ v + a u_n^- v + t \phi_1 v) dx. \]

Dividing by \(||u_n||\), using the fact \(\{D\tilde{I}(v_n)\}_1^\infty\) is bounded, and letting \(n \to \infty\), we can obtain

\[ \int_\Omega (\nabla v \cdot \nabla v - b w^+ v + a w^- v) dx = 0. \]  \hspace{1cm} (3.12)

Since (3.11) holds for arbitrary \(z \in W\) and (3.12) holds for arbitrary \(v \in V\) and \(H\) is direct sum of \(V\) and \(W\), we conclude that

\[ \int_\Omega (\nabla w \cdot \nabla y - b w^+ y + a w^- y) dx = 0 \quad \text{for all} \quad y \in H. \]

By (3.3), \(w\) is a solution of

\[ A w + b w^+ - a w^- = 0, \quad w|_{\partial \Omega} = 0. \]  \hspace{1cm} (3.13)

Since \(||w|| = 1\), this contradicts the assumption that (3.13) has only the trivial solution (cf. [9]). Hence the sequence \(\{V_n\}_1^\infty\) is bounded and the lemma is proved. \(\square\)

Let \(\hat{V}\) be the vector space spanned by an eigenfunction \(\phi_2\). Let \(\hat{W}\) denote the orthogonal complement of \(\hat{V}\) and let \(\hat{P} : H \to \hat{V}\) denote the orthogonal projection of \(H\) onto \(\hat{V}\). By the use of (3.1), (3.2) and Theorem 3.1, we have the following statements.

Given \(\hat{v} \in \hat{V}\) and \(t \in \mathbb{R}\), there exists a unique solution \(\hat{z} = \hat{\theta}(\hat{v})\) of

\[ A\hat{z} + (I - \hat{P})g(\hat{v} + \hat{z}) = t \phi_1, \quad \hat{z}|_{\partial \Omega} = 0, \]

where \(\hat{z} \in \hat{W}\).
If \( \hat{z} = \hat{\theta}(\hat{v}) \), then \( \hat{\theta} \) is continuous on \( \hat{V} \). Let \( \hat{F}_0(\hat{v}) \) denote the functional defined by \( \hat{F}_0(\hat{v}) = F(\hat{v} + \hat{\theta}(\hat{v})) \). Then \( \hat{F}_0 \) has a continuous Fréchet derivative \( D\hat{F}_0 \) with respect to \( \hat{v} \) and \( u \) is a solution of equation (3.1) if and only if \( u = \hat{v} + \hat{\theta}(\hat{v}) \) and \( D\hat{F}_0(\hat{v}) = 0 \), where \( \hat{v} = \hat{P}u \). By Theorem 3.3, for each fixed \( t \) the functional \( \hat{F}_0 \) satisfies the Palais-Smale condition.

By Theorem 3.1, the functional \( \hat{F}_0(\hat{v}) \) satisfy the following lemma.

**Lemma 3.4.** If \( t > 0 \) there exists \( \alpha = \alpha(t) > 0 \) such that if \( \hat{v} \in \hat{V} \) and \( \|\hat{v}\|_0 < \alpha(t) \), then \( \hat{\theta}(\hat{v}) = t\phi_1/(b - \lambda_1) \) for \( t > 0 \) and the point \( \hat{v} = 0 \) is a stric local point of maximum for \( \hat{F}_0 \).

**Lemma 3.5.** For \( k > 0 \) and \( t = 0 \), \( \hat{F}_0(k\hat{v}) = k^2\hat{F}_0(\hat{v}) \).

**Proof.** Since \( g \) is positively homogeneous of degree one, it follows that if \( \hat{v} \in \hat{V}, \hat{z} \in \hat{W} \) and \( A\hat{z} + (I - \hat{P})g(\hat{v} + \hat{z}) = 0, \hat{z}|_{\partial\Omega} = 0 \), then \( A(k\hat{z}) + (I - \hat{P})g(k\hat{v} + k\hat{z}) = 0 \). Therefore, \( \hat{\theta}(k\hat{v}) = k\hat{\theta}(\hat{v}) \). We see that \( F_0(ku) = k^2F(u) \) for \( u \in H \) and \( k > 0 \). Hence, \( \hat{F}_0(k\hat{v}) = F(k\hat{v} + \hat{\theta}(k\hat{v})) = k^2F(\hat{v} + \hat{\theta}(\hat{v})) = k^2\hat{F}_0(\hat{v}) \).

**Lemma 3.6.** Let \( \lambda_1 < a < \lambda_2 < b < \lambda_3 \). Then we have:

(a) For \( t = 0 \), \( \hat{F}_0(\hat{v}) > 0 \) for all \( \hat{v} \in \hat{V} \) with \( \hat{v} \neq 0 \).

(b) For \( t > 0 \), \( \hat{F}_0(\hat{v}) \to \infty \) as \( \|\hat{v}\|_0 \to \infty \).

(c) For fixed \( t > 0 \), \( \hat{F}_0(\hat{v}) \to \infty \) along a \( \phi_2 \)-axis.

**Proof.** With Lemma 3.5 and [7], we have (a) and (b).

(c) For fixed \( t \) we see that \( F(\hat{v} + \hat{\theta}(\hat{v})) = F(v + \theta(v)) \). Let \( \hat{F}|_{\hat{V}} \) be the restriction of \( \hat{F} \) to the \( \hat{V} \). Then \( \hat{F}|_{\hat{V}} \to \hat{F}_0 \). By (b), if \( t > 0 \), then \( \hat{F}(v) \to \infty \) as along a \( \phi_2 \)-axis.

**Lemma 3.7.** Let \( \lambda_1 < a < \lambda_2 < b < \lambda_3 \) and \( t = b - \lambda_1 \) and \( q^2 | \lambda_2 - a | > | \lambda_1 - a | \).

Then we have \( \hat{F}(v) \to +\infty \) as \( \|v\|_0 \to \infty \) along a boundary ray of \( C_3 \).

**Proof.** Let \( v = v_p + v_s \in C_3 \) and \( v_s = c_1\phi_1 + c_2\phi_2 \). Then we have

\[
\hat{F}(v) = \int_\Omega \left[ \frac{1}{2}(|\nabla (v_p + v_s)|^2 - a((v_p + v_s)^r)^2) + (b - \lambda_1)\phi_1(v_p + v_s) \right] dx.
\]

We note that \( v_p + v_s \in \partial C_3 \) if and only if \( c_2 = q(c_1 + 1), c_1 \leq -1 \). It can be shown easily the following holds

\[
\hat{F}(v) = \frac{1}{2}((\lambda_1 - a)c_1^2 + q^2(\lambda_2 - a)c_1^2) + (q^2(\lambda_2 - a) + (b - a))c_1 + \frac{1}{2}((\lambda_2 - a)q^2 + (b - a)) + C,
\]

where \( C = \int_\Omega \left[ \frac{1}{2}(|\nabla v_p|^2 - bu_p^2) + (b - \lambda_1)\phi_1 v_p \right] dx \). Hence if \( v \in \partial C_3 \), then we have \( \hat{F}(v) \to +\infty \) as \( c_1 \to -\infty \).

**Theorem 3.8.** Let \( \lambda_1 < a < \lambda_2 < b < \lambda_3 \) and \( t = b - \lambda_1 \). Then \( \hat{F}(v) \) has a critical point in \( \text{Int}C_1 \), and at least one critical point in \( \text{Int}C_2 \), and at least one critical point in \( \text{Int}C_4 \).

**Proof.** We denote that \( -\hat{F}(v) = \hat{F}(v) \). By Theorem 3.2 (a), if \( t = b - \lambda_1 \), then there exists a small open neighborhood \( B_p \) of \( v_p \) in \( C_1 \) such that in \( B_p, v_p = \phi_1 \) is a
strict local point of maximum for $\tilde{F}(v)$. Hence $v_p$ is a stric local point of minimum for $\tilde{F}_*(v)$ in $C_1$. By Lemma 3.6 (c), $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a $\phi_2$-axis. and $\tilde{F}_* \in C^1(V, \mathbb{R})$ satisfies the Palais-Smale condition.

Since $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a $\phi_2$-axis, we can choose $v_0$ on $\phi_2$-axis such that $\tilde{F}_*(v_0) < \tilde{F}_*(v_p)$. Let $\Gamma$ be the set of all paths in $V$ joining $v_p$ and $v_0$. We write

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_*(v).$$

The fact that in $B_p$, $v_p$ is a strict local point of minimum of $\tilde{F}_*$, the fact that $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a $\phi_2$-axis, the fact $\tilde{F}_*$ satisfies the Palais-Smale condition, and the Mountain Pass Theorem imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_*(v)$$

is a critical value of $\tilde{F}_*$ (see Mountain Pass Theorem and [3, 9]). When $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1$, equation (3.1) has a unique positive solution $v_p$ and no negative solution. Hence there exists a critical point $v_3$, in $\text{Int}(C_2 \cup C_4)$, of $\tilde{F}_*$ such that

$$\tilde{F}_*(v_3) = c.$$ 

We prove that if $v_3 \in \text{Int}C_4$ such that $\tilde{F}_*(v_3) = c$, then there exists another critical point $v \in \text{Int}C_2$ of $\tilde{F}_*$. Suppose $v_3 \in \text{Int}C_4$. Since $\tilde{F}_*(v) \to -\infty$ as $\|v\|_0 \to \infty$ along a $\phi_2$-axis, we can choose $v_1$ on this $\phi_2$-axis such that $\tilde{F}_*(v_1) < \tilde{F}_*(v_p)$. Let $\Gamma_1$ be the set of all paths in $C_1 \cup C_2 \cup C_3$ joining $v_p$ and $v_1$. We write

$$c' = \inf_{\gamma \in \Gamma_1} \sup_{\gamma} \tilde{F}_*(v).$$

We note that $\tilde{F}_*(v) \to \infty$ as $\|v\|_0 \to \infty$ along a negative $\phi_1$-axis or along a boundary ray, $c_2 = q(c_1 + 1)(c_1 \geq -1)$, of $C_1$, where $v = v_0 + c_1 \phi_1 + c_2 \phi_2 \in \partial C_1$.

Let us fix $\varepsilon, \eta$ as in Deformation Lemma with $E = V, F = \tilde{F}_*, c = c', K_{\varepsilon} = \phi$ and taking $\varepsilon < \frac{1}{2}(c' - \tilde{F}_*(v_p))$. Taking $\gamma \in \Gamma_1$ such that $\sup_{\gamma} \tilde{F}_* \leq c'$. From Deformation Lemma (see [3]), $\eta(1, \cdot) \circ \gamma \in \Gamma_1$ and

$$\sup_{\gamma} \tilde{F}_*(\eta(1, \cdot) \circ \gamma) \leq c' - \varepsilon < c',$$

which is a contradiction. Therefore there exists a critical point $v_4$ of $\tilde{F}_*$ at level $c'$ such that $v_4 \in C_1 \cup C_2 \cup C_3$ and $\tilde{F}_*(v_4) = c'$. Since equation (3.1) has a unique positive solution $v_p$ and no negative solution when $\lambda_1 < a < \lambda_2 < b < \lambda_3$ and $t = b - \lambda_1(> 0)$, the critical point $v_4$ belongs to $\text{Int}C_2$.

Similarly, we have that if $v_3 \in \text{Int}C_2$ with $\tilde{F}_*(v_3) = c$, then $\tilde{F}_*(v)$ has another critical point in $\text{Int}C_4$. The critical point of $\tilde{F}_*$ if and only if the critical point of $\tilde{F}$. Hence this completes the theorem.

**Theorem 3.9.** Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. For $1 \leq i \leq 4$, let $\Pi(C_i) = R_i$. Then $R_2 = R_1 \cup R_4^*$ and $R_4 = R_1 \cup R_2^*$. 


Proof. Let $h \in V$. We note that $v$ is a solution of the equation
\[ \Pi(v) = Av + P(b(v + \theta(v))^+ - a(v + \theta(v))^-) = h \quad \text{in} \quad V \]
if and only if $v$ is a critical point of $\bar{F}$. Hence it follows from Theorem 3.8 that $R_2 \cap R_1 \neq \emptyset$. Since $R_2$ is one of sets $R_1 \cup R_1^*$ or $R_3 \cup R_2^*$, $R_2$ must be $R_1 \cup R_1^*$.

On the other hand, it follows from Theorem 3.8 that $R_4 \cap R_1 \neq \emptyset$. Since $R_4$ is one of sets $R_1 \cup R_2^*$ or $R_3 \cup R_4^*$, $R_4$ must be $R_1 \cup R_2^*$.

By Theorem 2.2, Theorem 2.3, Theorem 2.4 and Theorem 3.9, we obtain the main theorem of the equation (1.2).

Theorem 3.9. Let $\lambda_1 < a < \lambda_2 < b < \lambda_3$. Then we have the following:

(a) If $h \in \text{Int}R_1$, then equation (1.2) has a positive solution and at least two change sign solutions.

(b) If $h \in \partial R_1$, then equation (1.2) has a positive solution and at least one change sign solution.

(c) If $h \in \text{Int}R_i^* (i = 2, 4)$, then equation (1.2) has at least one change sign solution.

(d) If $h \in \text{Int}R_3$, then equation (1.2) has only the negative solution.

(e) If $h \in \partial R_3$, then equation (1.2) has a negative solution.

References


