FRACTIONAL MULTIVARIATE OPIAL TYPE INEQUALITIES OVER SPHERICAL SHELLS

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ABSTRACT: Here is introduced the concept of multivariate fractional differentiation especially of the fractional radial differentiation, by extending the univariate definition of Canavati [11]. Then we produce Opial type inequalities over compact and convex subsets of \( \mathbb{R}^N \), \( N \geq 2 \), mainly over spherical shells, studying the problem in all possibilities. Our results involve one, or two, or more functions.

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1. INTRODUCTION

This work is motivated by the articles of Opial [12], Beesack [10], and Anastassiou [3]-[9].

We would like to mention Theorem 1.1.

**Theorem 1.1.** (see Opial [12], 1960) Let \( c > 0 \), and \( y(x) \) be real, continuously differentiable on \([0, c]\), with \( y(0) = y(c) = 0 \). Then

\[
\int_0^c |y(x)y'(x)|\,dx \leq \frac{c}{4} \int_0^c (y'(x))^2\,dx.
\]

Equality holds for the function \( y(x) = x \) on \([0, c/2]\), and \( y(x) = c - x \) on \([c/2, c]\).

The next result implies Theorem 1.1 and is used a lot in applications.

**Theorem 1.2.** (see Beesack [10], 1962) Let \( b > 0 \), If \( y(x) \) is real, continuously differentiable on \([0, b]\), and \( y(0) = 0 \) then

\[
\int_0^b |y(x)y'(x)|\,dx \leq \frac{b}{2} \int_0^b (y'(x))^2\,dx.
\]

Equality holds only for \( y = mx \), where \( m \) is a constant.

We describe here our specific multivariate setting. Let the balls \( B(0, R_1), B(0, R_2) \); \( 0 < R_1 < R_2 \). Here \( B(0, R) := \{ x \in \mathbb{R}^N : |x| < R \} \subseteq \mathbb{R}^N \), \( N \geq 2 \), \( R > 0 \), and...
We consider here the space \( S^{N-1} := \{ x \in \mathbb{R}^N : |x| = 1 \} \), where \(| \cdot |\) is the Euclidean norm. Let \( d\omega \) be the element of surface measure on \( S^{N-1} \) and let \( \omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)} \).

For \( x \in \mathbb{R}^N \setminus \{ 0 \} \) we can write uniquely \( x = r\omega \), where \( r = |x| > 0 \), and \( \omega = \frac{x}{r} \in S^{N-1} \), \(|\omega| = 1\).

Let the *spherical shell* \( A := B(0, R_2) - \overline{B(0, R_1)} \). We have that \( \text{Vol}(A) = \frac{\omega_N(R_2^N - R_1^N)}{N} \). Indeed \( \hat{A} = [R_1, R_2] \times S^{N-1} \).

For \( F \in C(\hat{A}) \) it holds

\[
\int_A F(x)dx = \int_{S^{N-1}} \left( \int_{R_1}^{R_2} F(r\omega)r^{N-1}dr \right) d\omega,
\]

we exploit a lot this formula here.

In this article we present a series of various fractional multivariate Opial type inequalities over spherical shells and arbitrary domains. Opial type inequalities find applications in establishing uniqueness of solution of initial value problems for differential equations and their systems, see Willett [13].

2. RESULTS

We make

**Remark 2.1.** We introduce here the *partial fractional derivatives*. Let \( f : [0, 1]^2 \to \mathbb{R} \). Let \( \nu > 0 \), \( n := [\nu] \), \( \alpha := \nu - n \), \( 0 < \alpha < 1 \); \( \mu > 0 \), \( m := [\mu] \), \( \beta := \mu - m \), \( 0 < \beta < 1 \). Assume \( \exists \frac{\partial^{n+m}f(t, s)}{\partial x^n \partial y^m} \in C([0, 1]^2) \), then \( (x-t)^{-\alpha}(y-s)^{-\beta} \frac{\partial^{n+m}f(t, s)}{\partial x^n \partial y^m} \) is integrable over \([0, x] \times [0, y]; x, y \in [0, 1] \), that is

\[
F(x, y) := \int_0^x \int_0^y (x-t)^{-\alpha}(y-s)^{-\beta} \frac{\partial^{n+m}f(t, s)}{\partial x^n \partial y^m} dt ds
\]

is real valued.

Thus, by Fubini’s Theorem, the order of integration in (2.1) does not matter.

Let now \( g \in C([0, 1]) \), we define the *Riemann-Liouville integral*, \( \Gamma \) is the gamma function: \( \Gamma(\nu) := \int_0^\infty e^{-t}t^{\nu-1}dt \), as

\[
(J_\nu g)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1}g(t)dt, \quad 0 \leq x \leq 1.
\]

We consider here the space

\[
C^\nu([0, 1]) := \{ g \in C^n([0, 1]) : J_{1-\alpha} g^{(n)} \in C^1([0, 1]) \},
\]

then the \( \nu \)-*fractional derivative of* \( g \) is defined by \( g^{(\nu)} := (J_{1-\alpha} g^{(n)})' \), see Canavati [11].

We assume here \( f(\cdot, y) \in C^\nu([0, 1]), \forall y \in [0, 1] \), then we define the \( \nu \)-*partial fractional derivative* of \( f \) with respect to \( x \) \( \frac{\partial^\nu f(x, y)}{\partial x^\nu} \) as

\[
\frac{\partial^\nu f(x, y)}{\partial x^\nu} := \frac{\partial}{\partial x} \left( J_{1-\alpha} \frac{\partial^n f(x, y)}{\partial x^n} \right), \quad \forall (x, y) \in [0, 1]^2.
\]
Also, we assume \( f(x, \cdot) \in C^\mu([0, 1]), \forall x \in [0, 1], \) where
\[
C^\mu([0, 1]) := \{ g \in C^m([0, 1]) : J_{1-\beta}g^{(m)} \in C^1([0, 1]) \}. \tag{2.5}
\]

Then we define the \( \mu \)-partial fractional derivative of \( f \) with respect to \( y \): \( \frac{\partial f^\mu}{\partial y^\mu}(x, \cdot) \)
as
\[
\frac{\partial f^\mu(x, y)}{\partial y^\mu} := \frac{\partial}{\partial y} \left( J_{1-\beta} \frac{\partial f^m}{\partial y^m}(x, y) \right), \quad \forall (x, y) \in [0, 1]^2. \tag{2.6}
\]

Define the space
\[
C^{\nu+\mu}([0, 1]^2) := \{ f \in C^{n+m}([0, 1]^2) :
J_{1-\alpha} \left( \frac{\partial^n f(\cdot, y)}{\partial x^n} \right) \in C^1([0, 1]), \forall y \in [0, 1];
J_{1-\beta} \left( \frac{\partial^m f(x, \cdot)}{\partial x^m} \right) \in C^1([0, 1]), \forall x \in [0, 1];
\exists F_x, F_y, F_{yx} \in C([0, 1]^2) \}. \tag{2.7}
\]

Define the mixed fractional partial derivative:
\[
\frac{\partial^{\nu+\mu} f(x, y)}{\partial x^\nu \partial y^\mu} := \frac{1}{\Gamma(1-\alpha) \Gamma(1-\beta)} \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y (x-t)^{-\alpha}(y-s)^{-\beta} \frac{\partial^{n+m} f(t, s)}{\partial x^n \partial y^m} dt ds. \tag{2.8}
\]

One can have anchor points \( x_0, y_0 \neq 0 \), then all above definitions go through for \( x \geq x_0, y \geq y_0 \).

**Conclusion 1.** Clearly then we have \( F_{xy} = F_{yx} \), and
\[
\frac{\partial^{\nu+\mu} f}{\partial x^\nu \partial y^\mu} = \frac{\partial^{\nu+\mu} f}{\partial y^\nu \partial x^\mu}. \tag{2.9}
\]

So the order of fractional differentiation is immaterial.

Here, it is by definition
\[
\frac{\partial^{\nu+\mu} f(x, y)}{\partial y^\mu \partial x^\nu} := \frac{1}{\Gamma(1-\alpha) \Gamma(1-\beta)} \frac{\partial^2}{\partial y \partial x} \int_0^x \int_0^y (x-t)^{-\alpha}(y-s)^{-\beta} \frac{\partial^{n+m} f(t, s)}{\partial y^n \partial x^m} dt ds. \tag{2.10}
\]

**Comments.** 1) Let \( \nu = 0 \), then \( n = \alpha = 0 \), and (2.8) becomes
\[
\frac{\partial^\mu f(x, y)}{\partial y^\mu} = \frac{1}{\Gamma(1-\beta)} \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y (y-s)^{-\beta} \frac{\partial^m f(t, s)}{\partial y^m} dt ds
= \frac{1}{\Gamma(1-\beta)} \frac{\partial^2}{\partial y \partial x} \int_0^x \int_0^y (y-s)^{-\beta} \frac{\partial^m f(t, s)}{\partial y^m} dt ds
= \frac{1}{\Gamma(1-\beta)} \left( \frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \int_0^x (y-s)^{-\beta} \frac{\partial^m f(t, s)}{\partial y^m} dt ds \right) \right) =: (*). \tag{2.11}
\]
Notice for fixed \( y \) we have that \( (y - s)^{-\beta} \frac{\partial^m f(t, s)}{\partial y^m} \) is integrable over \( [0, y] \), so the function

\[
\varphi(t) := \int_0^y (y - s)^{-\beta} \frac{\partial^m f(t, s)}{\partial y^m} \, ds
\]

(2.12)
is real valued for any \( t \in [0, x] \).

By continuity of \( \frac{\partial^m f}{\partial y^m} \) we have true that \( \forall \varepsilon > 0 \) \( \exists \delta > 0 \) : whenever \( |t_1 - t_2| < \delta \) we have

\[
\left| \frac{\partial^m f(t_1, s)}{\partial y^m} - \frac{\partial^m f(t_2, s)}{\partial y^m} \right| < \varepsilon.
\]

We further have

\[
\varphi(t_1) - \varphi(t_2) = \int_0^y (y - s)^{-\beta} \left( \frac{\partial^m f(t_1, s)}{\partial y^m} - \frac{\partial^m f(t_2, s)}{\partial y^m} \right) \, ds.
\]

Hence

\[
|\varphi(t_1) - \varphi(t_2)| \leq \int_0^y (y - s)^{-\beta} \left| \frac{\partial^m f(t_1, s)}{\partial y^m} - \frac{\partial^m f(t_2, s)}{\partial y^m} \right| \, ds
\]

\[
\leq \varepsilon \int_0^y (y - s)^{-\beta} \, ds = \frac{\varepsilon y^{1-\beta}}{1-\beta},
\]

(2.13)

proving \( \varphi(t) \) is continuous.

Consequently

\[
(\star) = \frac{1}{\Gamma(1 - \beta)} \left( \frac{\partial}{\partial y} \left( \int_0^y (y - s)^{-\beta} \frac{\partial^m f(x, s, t; y)}{\partial y^m} \, ds \right) \right) =: \frac{\partial^m f(x, y)}{\partial y^m}.
\]

(2.14)

**Conclusion 2.** When \( \nu = 0 \), the fractional mixed partial derivative collapses to the single fractional partial derivative.

2) Let \( \mu = 0 \), then \( m = \beta = 0 \), and (2.8) becomes

\[
\frac{\partial^\nu f(x, y)}{\partial x^\nu} = \frac{1}{\Gamma(1 - \alpha)} \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y (x - t)^{-\alpha} \frac{\partial^m f(t, s)}{\partial x^m} \, dt \, ds
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \left( \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \int_0^y \left( \int_0^x (x - t)^{-\alpha} \frac{\partial^m f(t, s)}{\partial x^m} \, dt \right) \, ds \right) \right) \right)
\]

(2.15)

( notice \( \int_0^x (x - t)^{-\alpha} \frac{\partial^m f(t, s)}{\partial x^m} \, dt \) is continuous in \( s \in [0, y] \))

\[
= \frac{1}{\Gamma(1 - \alpha)} \left( \frac{\partial}{\partial x} \left( \int_0^x (x - t)^{-\alpha} \frac{\partial^m f(t, y)}{\partial x^m} \, dt \right) \right) =: \frac{\partial^\nu f(x, y)}{\partial x^\nu}.
\]

(2.16)

**Conclusion 3.** When \( \mu = 0 \), the mixed fractional derivative collapses again to the single one.

3) Let now \( n = \nu \in \mathbb{N} \), i.e. \( \alpha = 0 \), then

\[
\frac{\partial^\nu f(x, y)}{\partial x^\nu} = \frac{\partial}{\partial x} \left( \int_0^x \frac{\partial^\nu f(t, y)}{\partial x^\nu} \, dt \right) = \frac{\partial^n f(x, y)}{\partial x^n},
\]

(2.17)

the ordinary one.

4) When \( m = \mu \in \mathbb{N} \), i.e. \( \beta = 0 \), then

\[
\frac{\partial^\mu f(x, y)}{\partial y^\mu} = \frac{\partial}{\partial y} \int_0^y \frac{\partial^\mu f(x, s)}{\partial y^\mu} \, ds = \frac{\partial^m f(x, y)}{\partial y^m},
\]

(2.18)

the ordinary one.
5) Furthermore, let finally both \( \nu = n \in \mathbb{N} \) and \( \mu = m \in \mathbb{N} \), i.e. \( \alpha = \beta = 0 \). Then

\[
\frac{\partial^{\nu+\mu} f(x, y)}{\partial x^{\nu} \partial y^{\mu}} = \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y \frac{\partial^{n+m} f(t, s)}{\partial x^n \partial y^m} \, dt \, ds
\]

\[
= \frac{\partial}{\partial x} \left( \frac{\partial}{\partial y} \left( \int_0^y \left( \int_0^x \frac{\partial^{n+m} f(t, s)}{\partial x^n \partial y^m} \, dt \right) \, ds \right) \right)
\]

\[
= \frac{\partial}{\partial x} \left( \int_0^x \frac{\partial^{n+m} f(t, y)}{\partial x^n \partial y^m} \, dt \right) = \frac{\partial^{n+m} f(x, y)}{\partial x^n \partial y^m},
\]

proving the that fractional mixed partial collapses to the ordinary one. Fractional differentiation is a linear operation.

**Conclusion 4.** The above definitions we gave for the fractional partial derivatives are natural extensions of the ordinary positive integer ones.

Having introduced the fractional partial derivatives we are ready to develop our Opial type results.

We make

**Remark 2.2.** First we consider a general domain. Let \( Q \) be a compact and convex subset of \( \mathbb{R}^N \), \( N \geq 2 \); \( z := (z_1, \ldots, z_N) \), \( x_0 := (x_{01}, \ldots, x_{0N}) \in Q \) be fixed. Let \( f \in C^n(Q) \), \( n \in \mathbb{N} \). Set \( g_z(t) = f(x_0 + t(z - x_0)) \), \( 0 \leq t \leq 1 \);

\[
g_z(0) = f(x_0), \quad g_z(1) = f(z).
\]

Then it holds

\[
g_z^{(j)}(t) = \left( \sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f(x_0 + t(z - x_0)),
\]

where \( j = 0, 1, 2, \ldots, n \), and in particular

\[
g_z'(t) = \sum_{i=1}^N (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)),
\]

\( 0 \leq t \leq 1 \).

Clearly here \( g_z \in C^n([0, 1]). \) Let first \( 1 \leq \nu < 2 \), in that case we take \( n := [\nu] = 1 \). Following Anastassiou [2] and by assuming that as function of \( t : f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0, 1]) \), \( i = 1, \ldots, N \), then there exists \( g_z^{(\nu)} = (\mathcal{J}_{2-\nu} g_z')(\nu) \), and it holds

\[
g_z^{(\nu)}(t) = \sum_{i=1}^N (z_i - x_{0i}) \left( \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)) \right)^{(\nu-1)};
\]

\( 0 \leq t \leq 1 \).

Also here we have

\[
(\mathcal{J}_{2-\nu} g_z')(t) = \frac{\sum_{i=1}^N (z_i - x_{0i})}{\Gamma(2 - \nu)} \int_0^t (t - s)^{1-\nu} f_{x_i}(x_0 + s(z - x_0)) \, ds,
\]

\( 0 \leq t \leq 1 \).
Remark 2.2.Obviously \((\mathcal{J}_{2-\nu} g_z^n(t)) \in C^1([0, 1])\) and \((\mathcal{J}_{2-\nu} g_z^n)(0) = 0\). Therefore by (1.2) we get
\[
\int_0^s |\mathcal{J}_{2-\nu} g_z^n(t)| \, |D^\nu g_z(t)| \, dt \leq \frac{s}{2} \int_0^s (D^\nu g_z(t))^2 \, dt, \quad \forall s \in [0, 1].
\] (2.24)

We have established the following Opial type result.

**Theorem 2.1.** Let \(Q\) be a compact and convex subset of \(\mathbb{R}^N, N \geq 2\); \(z, x_0 \in Q\) be fixed; \(1 \leq \nu < 2\). Let \(f \in C^1(Q)\). Assume that as a function of \(t : f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0, 1]), \, i = 1, \ldots, N\).

Then
\[
\frac{1}{\Gamma(2-\nu)} \int_0^s \left| \sum_{i=1}^N (z_i - x_{0i}) \left( \int_0^t (t-s)^{1-\nu} f_{x_i}(x_0 + s(z-x_0)) \, ds \right) \right|
\]
\[
\sum_{i=1}^N (z_i - x_{0i})(f_{x_i}(x_0 + t(z-x_0)))^{(\nu-1)} \, dt
\]
\[
\leq \frac{s}{2} \int_0^s \left( \sum_{i=1}^N (z_i - x_{0i})(f_{x_i}(x_0 + t(z-x_0)))^{(\nu-1)} \right)^2 \, dt,
\] (2.25)
\(
\forall s \in [0, 1].
\)

**Remark 2.2.** (Continuation) Let here \(\nu \geq 2\) and \(n := [\nu], \, \beta := \nu - n\). We assume that as functions of \(t : f_{\alpha}(x_0 + t(z - x_0)) \in C^{(\nu-n)}([0, 1]), \) for all \(\alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+, \, i = 1, \ldots, N; \, |\alpha| := \sum_{i=1}^N \alpha_i = n\). Clearly then there exists \(g_z^{(n)} = (\mathcal{J}_{1-\beta} g_z^n)^{\nu}\), and it holds
\[
g_z^{(n)}(t) = \left[ \left( \sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right]^{(\nu-n)} (x_0 + t(z-x_0)),
\] (2.26)
all \(t \in [0, 1]\).

Of course, it holds
\[
(\mathcal{J}_{1-\beta} g_z^n(t))^2 \equiv \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}
\]
\[
\left\{ \left[ \left( \sum_{i=1}^N (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + s(z-x_0)) \right\} \, ds.
\] (2.27)

Notice \((\mathcal{J}_{1-\beta} g_z^n(0)) = 0\). Hence again by (1.2) we get
\[
\int_0^s |\mathcal{J}_{1-\beta} g_z^n(t)| \, |D^\nu g_z(t)| \, dt \leq \frac{s}{2} \int_0^s (D^\nu g_z(t))^2 \, dt, \quad \forall s \in [0, 1].
\] (2.28)

We have proved the following general Opial type of result.

**Theorem 2.2.** Let \(Q\) be a compact and convex subset of \(\mathbb{R}^N, N \geq 2\); \(z, x_0 \in Q\) be fixed; \(\nu \geq 2, \, n := [\nu], \, \beta := \nu - n\). Let \(f \in C^n(Q)\). Assume that as a function
of \( t : f_\alpha(x_0 + t(z - x_0)) \in C^{(\nu-n)}([0,1]), \) for all \( \alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, N; \) \( |\alpha| := \sum_{i=1}^{N} \alpha_i = n. \) Then

\[
\frac{1}{\Gamma(1-\beta)} \int_0^s \left| \int_0^t (t-s)^{-\beta} \left\{ \left[ \left( \sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right](x_0 + s(z - x_0)) \right\} ds \right| \left[ \left( \sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right](x_0 + t(z - x_0)) \right| \right. dt \\
\leq \frac{s}{2} \int_0^s \left\{ \left[ \left( \sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right](x_0 + t(z - x_0)) \right\}^2 dt,
\]

\forall s \in [0, 1].

**Note.** Following the last pattern one can transfer any univariate Opial type inequality (see Agarwal and Pang [1]), into this fractional multivariate general setting. Since no chain rule is valid in the fractional differentiation, inequalities such as (2.28), (2.29) are not revealing themselves, to totally decompose into all of their ingredients. Next, working over *spherical shells* we obtain a series of various Opial type fractional multivariate inequalities that look nice and are very clear.

We give

**Definition 2.1.** (see Anastassiou [7] and Anastassiou [5], p. 540) In the following we carry earlier notions introduced in Remark 2.1, over to arbitrary \([a,b] \subseteq \mathbb{R}. \) Let \( x, x_0 \in [a,b] \) such that \( x \geq x_0, x_0 \) is fixed. Let \( f \in C([a,b]) \) and define

\[
(\mathcal{J}_\nu^x f)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \quad x_0 \leq x \leq b,
\]

the generalized Riemann-Liouville integral. We consider the subspace \( C_\nu^{x_0}([a,b]) \) of \( C^{\nu}([a,b]):\)

\[
C_\nu^{x_0}([a,b]) := \{ f \in C^{\nu}([a,b]) : \mathcal{J}_1 f = x_0 \}
\]

Hence, let \( f \in C_\nu^{x_0}([a,b]), \) we define the generalized \( \nu \)-fractional derivative of \( f \) over \([x_0,b] \) as

\[
D_\nu^{x_0} f := (\mathcal{J}_1 f^{(n)})'.
\]

Notice that

\[
(\mathcal{J}_1 f^{(n)})'(x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x-t)^{-\alpha} f^{(n)}(t) dt
\]

exists for \( f \in C_\nu^{x_0}([a,b]). \)

Next we use

**Theorem 2.3.** (see Anastassiou [8] and Anastassiou [5], p. 567) Let \( \gamma_i \geq 1, \nu \geq 2 \) such that \( \nu - \gamma_i \geq 1; \ i = 1, \ldots, l \) and \( f \in C_\nu^{x_0}([a,b]) \) with \( f^{(j)}(x_0) = 0, j = 0, 1, \ldots, n-1, n := [\nu]. \) Here \( x, x_0 \in [a,b] : x \geq x_0. \) Let \( q_1, q_2 > 0 \) continuous
Lemma 2.1. Let $r_i > 0 : \sum_{i=1}^l r_i = r$. Let $s_1, s'_1 > 1 : \frac{1}{s_1} + \frac{1}{s'_1} = 1$ and $s_2, s'_2 > 1 : \frac{1}{s_2} + \frac{1}{s'_2} = 1$ and $p > s_2$. Furthermore suppose that

$$Q_1 := \left( \int_{x_0}^x (q_1(\omega))^{s'_1} d\omega \right)^{1/s'_1} < +\infty$$  \hspace{1cm} (2.34)

and

$$Q_2 := \left( \int_{x_0}^x (q_2(\omega))^{-s_2/p} d\omega \right)^{r/s'_2} < +\infty.$$  \hspace{1cm} (2.35)

Call $\sigma := \frac{p-s_2}{ps_2}$. Then it holds

$$\int_{x_0}^x q_1(\omega) \prod_{i=1}^l (|D_{x_0}^{\gamma_i}(f)(\omega)|^r d\omega \leq Q_1 Q_2$$

and

$$\prod_{i=1}^l \left\{ \frac{\sigma^{r_i} \sigma}{(\Gamma(\nu - \gamma_i))^{r_i} (\nu - \gamma_i - 1 + \sigma)^{r_i} \sigma} \right\} \cdot \frac{(x-x_0)^{\sum_{i=1}^l (\nu - \gamma_i - 1) r_i + \sigma r + \frac{1}{s_1}}}{((\sum_{i=1}^l (\nu - \gamma_i - 1) r_i + r s_1 + 1)^{1/s_1}} \cdot \left( \int_{x_0}^x q_2(\omega) |D_{x_0}^{\nu}(f)(\omega)|^p d\omega \right)^{r/p}.$$  \hspace{1cm} (2.36)

We next work in the setting of spherical shells introduced in the Introduction.

We need

**Definition 2.2.** Let $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $f \in C^n(\bar{A})$, $A$ is a spherical shell. Assume that there exists $\frac{\partial_{R_1}^\alpha f(x)}{\partial r^\alpha} \in C(\bar{A})$, given by

$$\frac{\partial_{R_1}^\alpha f(x)}{\partial r^\alpha} := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial r} \left( \int_{R_1}^r (r-t)^{-\alpha} \partial^{\nu}(f(t\omega)) d\omega \right),$$  \hspace{1cm} (2.37)

where $x \in \bar{A}$, i.e. $x = r \omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We call $\frac{\partial_{R_1}^\alpha f}{\partial r^\alpha}$ the radial fractional derivative of $f$ of order $\nu$.

We need

**Lemma 2.1.** Let $\gamma \geq 1$, $\nu > 2$ such that $\nu - \gamma \geq 1$. Let $f \in C^n(\bar{A})$ and there exists $\frac{\partial_{R_1}^\gamma f(x)}{\partial r^\gamma} \in C(\bar{A})$, $x \in \bar{A}$, $A$ a spherical shell. Further assume that $\frac{\partial^j f(R_1 \omega)}{\partial r^j} = 0$, $j = 0, 1, \ldots, n-1$, $n := [\nu]$, $\forall \omega \in S^{N-1}$. Then there exists $\frac{\partial_{R_1}^\beta f(x)}{\partial r^\beta} \in C(\bar{A})$.

**Proof.** The assumption implies that $\frac{\partial_{R_1}^\beta f(r\omega)}{\partial r^\beta} \in C([R_1, R_2]), \forall \omega \in S^{N-1}$, i.e. $f(r\omega) \in C_{R_1}^\nu([R_1, R_2])$, $\forall \omega \in S^{N-1}$. Following Anastassiou [7], and Anastassiou [5], pp. 544-545, we get that there exists $\frac{\partial_{R_1}^\gamma f(r\omega)}{\partial r^\gamma}$ and is given by

$$\frac{\partial_{R_1}^\gamma f(r\omega)}{\partial r^\gamma} = \frac{1}{\Gamma(\nu - \gamma)} \int_{R_1}^r (r-t)^{\nu-\gamma-1} \frac{\partial^\nu f(t\omega)}{\partial r^\nu} dt, \hspace{1cm} (2.38)$$

where $x \in \bar{A}$, i.e. $x = r \omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.
indeed \( f(r \omega) \in C_{R_1}^\nu([R_1, R_2]), \forall \omega \in S^{N-1}. \)

Hence
\[
\frac{\partial_{R_1}^\nu f(r \omega)}{\partial r^{\nu - 1}} = \frac{1}{\Gamma(\nu - \gamma)} \int_{R_1}^{R_2} \mathcal{X}_{[R_1, r]}(t)(r - t)^{\nu - \gamma - 1} \frac{\partial_{R_1}^\nu f(t \omega)}{\partial r^{\nu - 1}} \, dt. \tag{2.39}
\]

Let \( r_n \to r, \omega_n \to \omega, \) then \( \mathcal{X}_{[R_1, r_n]}(t) \to \mathcal{X}_{[R_1, r]}(t), \) a.e. also \( (r_n - t)^{\nu - \gamma - 1} \to (r - t)^{\nu - \gamma - 1}, \) and
\[
\frac{\partial_{R_1}^\nu f(t \omega_n)}{\partial r^{\nu - 1}} \to \frac{\partial_{R_1}^\nu f(t \omega)}{\partial r^{\nu - 1}}.
\]

Furthermore it holds that
\[
\mathcal{X}_{[R_1, r_n]}(t)(r_n - t)^{\nu - \gamma - 1} \frac{\partial_{R_1}^\nu f(t \omega_n)}{\partial r^{\nu - 1}} \to \\
\mathcal{X}_{[R_1, r]}(t)(r - t)^{\nu - \gamma - 1} \frac{\partial_{R_1}^\nu f(t \omega)}{\partial r^{\nu - 1}}, \text{ a.e. on } [R_1, R_2]. \tag{2.40}
\]

However we have
\[
\mathcal{X}_{[R_1, r_n]}(t)(r_n - t)^{\nu - \gamma - 1} \left| \frac{\partial_{R_1}^\nu f(t \omega_n)}{\partial r^{\nu - 1}} \right| \\
\leq (R_2 - R_1)^{\nu - \gamma - 1} \left\| \frac{\partial_{R_1}^\nu f}{\partial r^{\nu - 1}} \right\|_\infty < \infty. \tag{2.41}
\]

Thus, by the Dominated convergence theorem we obtain
\[
\int_{R_1}^{R_2} \mathcal{X}_{[R_1, r_n]}(t)(r_n - t)^{\nu - \gamma - 1} \frac{\partial_{R_1}^\nu f(t \omega_n)}{\partial r^{\nu - 1}} \, dt \to \\
\int_{R_1}^{R_2} \mathcal{X}_{[R_1, r]}(t)(r - t)^{\nu - \gamma - 1} \frac{\partial_{R_1}^\nu f(t \omega)}{\partial r^{\nu - 1}} \, dt, \tag{2.42}
\]
proving the claim.

We present the very general result.

**Theorem 2.4.** Let \( \gamma_i \geq 1, \nu \geq 2, \) such that \( \nu - \gamma_i \geq 1; i = 1, \ldots, l, n := [\nu]. \) Let \( f \in C^n(A) \) and there exists \( \frac{\partial f}{\partial x^j}(x) \in C(A), x \in A, A \) is a spherical shell: \( A := B(0, R_2) - B(0, R_1) \subseteq \mathbb{R}^N, \) \( N \geq 2. \) Furthermore assume that \( \frac{\partial f}{\partial x^j}, j = 0, 1, \ldots, n - 1, \) vanish on \( \partial B(0, R_1). \) Let \( r_i > 0 : \sum_{i=1}^l r_i = p. \) Let \( s_1, s'_1 > 1 : \frac{1}{s_1} + \frac{1}{s'_1} = 1, \) and \( s_2, s'_2 > 1 : \frac{1}{s_2} + \frac{1}{s'_2} = 1, \) and \( p > s_2. \) Denote
\[
Q_1 = \left( \frac{R_2^{(N-1)s'_1+1} - R_1^{(N-1)s'_1+1}}{(N - 1)s'_1 + 1} \right)^{1/s'_1}, \tag{2.43}
\]
and
\[
Q_2 = \left( \frac{R_2^{(1-N)s'_2+1} - R_1^{(1-N)s'_2+1}}{(1 - N)s'_2 + 1} \right)^{p/s'_2}. \tag{2.44}
\]

Call \( \sigma := \frac{\nu - s_2}{p s_2}. \)
Also call

\[ C := Q_1 Q_2 \prod_{i=1}^{l} \frac{\sigma^{\rho_i}}{(\Gamma(\nu - \gamma_i))^{r_i}(\nu - \gamma_i - 1 + \sigma)^{r_i}} \]

\[ \frac{(R_2 - R_1)^{\sum_{i=1}^{l}(\nu - \gamma_i - r_i) + \frac{1}{s_2} - 1}}{(\sum_{i=1}^{l}(\nu - \gamma_i - 1)r_is_i) + s_1(\frac{p}{s_2} - 1) + 1} \]

Then

\[ \int_{A} \prod_{i=1}^{l} \left| \frac{\partial^\rho_i f(x)}{\partial^{r_i} x} \right|^{\sigma_i} \, dx \leq C \int_{A} \left| \frac{\partial^\nu f(x)}{\partial^{\nu} x} \right|^{p} \, dx. \tag{2.46} \]

**Proof.** The assumption imply that \( f(\omega) \in C^n([R_1, R_2]), \) \( \frac{\partial^\nu f(\omega)}{\partial^{\nu} \omega} \in C([R_1, R_2]), \) \( \forall \omega \in S^{n-1}. \) By Theorem 2.3 we have

\[ \int_{R_1}^{R_2} \int_{R_1}^{R_2} \prod_{i=1}^{l} \left| \frac{\partial^\nu_i f(\omega)}{\partial^{r_i} \omega} \right|^{\sigma_i} \, d\omega \]

\[ \leq C \int_{R_1}^{R_2} \int_{R_1}^{R_2} \prod_{i=1}^{l} \left| \frac{\partial^\nu_i f(\omega)}{\partial^{r_i} \omega} \right|^{\sigma_i} \, d\omega. \tag{2.47} \]

Therefore it holds

\[ \int_{S^{n-1}} \left( \int_{R_1}^{R_2} \int_{R_1}^{R_2} \prod_{i=1}^{l} \left| \frac{\partial^\nu_i f(\omega)}{\partial^{r_i} \omega} \right|^{\sigma_i} \, d\omega \right) \, d\omega \]

\[ \leq C \int_{S^{n-1}} \left( \int_{R_1}^{R_2} \int_{R_1}^{R_2} \prod_{i=1}^{l} \left| \frac{\partial^\nu_i f(\omega)}{\partial^{r_i} \omega} \right|^{\sigma_i} \, d\omega \right) \, d\omega. \tag{2.48} \]

Using Lemma 2.1 and by (1.3) we derive (2.46). \( \square \)

We mention

**Theorem 2.5.** (see Anastassiou [8] and Anastassiou [5], p. 573) Let \( \gamma_i \geq 1, \nu \geq 2 \) such that \( \nu - \gamma_i \geq 1; \, i = 1, \ldots, l \) and \( f \in C^\nu_{x_0}([a, b]) \) with \( f^{(j)}(x_0) = 0, \, j = 0, 1, \ldots, n-1, \, n := [\nu]. \) Here \( x, x_0 \in [a, b] : x \geq x_0. \) Let \( \bar{q}(w) \geq 0 \) continuous on \([a, b] \) and \( r_i > 0 : \sum_{i=1}^{l} r_i = r. \) Then it holds

\[ \int_{x_0}^{x} \bar{q}(w) \prod_{i=1}^{l} (|D^\nu_{x_0} f(w)|)^{r_i} \, dw \]

\[ \leq \left\{ \frac{\bar{q} \infty(|D^\nu_{x_0} f| \infty)^{r}}{\prod_{i=1}^{l} (\Gamma(\nu - \gamma_i + 1))^{r_i}} \right\} \cdot \left\{ \frac{(x - x_0)^{r - \sum_{i=1}^{l} r_i \gamma_i + 1}}{(r - \sum_{i=1}^{l} r_i \gamma_i + 1)} \right\}. \tag{2.49} \]

We give

**Theorem 2.6.** Let \( \gamma_i \geq 1, \nu \geq 2, \) such that \( \nu - \gamma_i \geq 1; \, i = 1, \ldots, l, \, n := [\nu]. \) Let \( f \in C^\nu(A) \) and there exists \( \frac{\partial^\nu_i f(x)}{\partial^{r_i} x} \in C(A), \, x \in A, \, A \) is a spherical shell: \( A := \)
\( B(0, R) - B(0, R_1) \subseteq \mathbb{R}^N \), \( N \geq 2 \). Furthermore assume that \( \frac{\partial f}{\partial r^j} \), \( j = 0, 1, \ldots, n-1 \), vanish on \( \partial B(0, R_1) \). Let \( r_i > 0 : \sum_{i=1}^l r_i = r \). Call
\[
M := \frac{R_2^{N-1}(R_2 - R_1)^{\nu - \sum_{i=1}^l r_i \gamma_i + 1}}{\prod_{i=1}^l (\Gamma(\nu - \gamma_i + 1))^{r_i}(\nu - \sum_{i=1}^l r_i \gamma_i + 1)} > 0. \tag{2.50}
\]
Then
\[
\int_A \left( \prod_{i=1}^l \left| \frac{\partial R_i f(x)}{\partial r^{\gamma_i}} \right|^{r_i} \right) dx \leq M \frac{2\pi^{N/2}}{\Gamma(N/2)} \left\| \frac{\partial R_i f}{\partial r^\nu} \right\|_{\infty, A}. \tag{2.51}
\]

**Proof.** By Theorem 2.5 we get that
\[
\int_{R_1}^{R_2} r^{N-1} \left( \prod_{i=1}^l \left| \frac{\partial R_i f(r \omega)}{\partial r^{\gamma_i}} \right|^{r_i} \right) dr \leq M \left\| \frac{\partial R_i f}{\partial r^\nu} \right\|_{\infty, A}. \tag{2.52}
\]
Hence it holds
\[
\int_{S^{N-1}} \left( \int_{R_1}^{R_2} r^{N-1} \left( \prod_{i=1}^l \left| \frac{\partial R_i f(r \omega)}{\partial r^{\gamma_i}} \right|^{r_i} \right) dr \right) d\omega \leq M \left( \omega_N \left\| \frac{\partial R_i f}{\partial r^\nu} \right\|_{\infty, A}. \tag{2.53}
\]
Using (1.3) and Lemma 2.1, we establish the claim. \( \Box \)

We need

**Theorem 2.7** (Anastassiou and Goldstein [9]). Let \( \gamma \geq 1 \), \( \nu \geq 2 \), \( \nu - \gamma \geq 1 \), \( \alpha, \beta > 0 \), \( r > \alpha \), \( r > 1 \); let \( p > 0 \), \( q > 0 \) be continuous functions on \( [a, b] \). Let \( f \in C_{x_0}^\nu([a,b]) \) with \( f^{(i)}(x_0) = 0 \), \( i = 0, 1, \ldots, n-1 \), \( n := [\nu] \). Let \( x, x_0 \in [a,b] \) with \( x \geq x_0 \). Then
\[
\int_{x_0}^{x} q(w) |D_x^\gamma f(w)|^\beta |D_x^\nu f(w)|^\alpha dw \leq K(p, q, \gamma, \nu, \alpha, \beta, r, x, x_0)
\]
\[
\cdot \left( \int_{x_0}^{x} p(w) |D_x^\nu f(w)|^r dw \right)^{(\alpha + \beta - \frac{\alpha r}{\nu})}. \tag{2.54}
\]
Here
\[
K(p, q, \gamma, \nu, \alpha, \beta, r, x, x_0) := \left( \frac{\alpha}{\alpha + \beta} \right)^{\alpha/r} \cdot \frac{1}{(\Gamma(\nu - \gamma))^{\beta}}
\]
\[
\cdot \left( \int_{x_0}^{x} (q(w))^{\frac{\gamma}{r - \alpha}} \cdot (p(w))^{-\alpha} \cdot (P_1(w))^{\frac{\beta (r-1)}{r-\alpha}} \cdot dw \right)^{-\frac{1}{r-\alpha}}, \tag{2.55}
\]
with
\[
P_1(w) := \int_{x_0}^{w} (p(t))^{-\frac{1}{\alpha - \nu - 1}} \cdot (w - t)^{(-\gamma - 1)(\frac{r-1}{r-\alpha})} dt. \tag{2.56}
\]

We present

**Theorem 2.8.** Let \( \gamma \geq 1 \), \( \nu \geq 2 \), \( n := [\nu] \), \( \nu - \gamma \geq 1 \), \( \alpha, \beta > 0 \), \( \alpha + \beta > 1 \). Let \( f \in C^\alpha(\bar{A}) \) and there exists \( \frac{\partial R_i f}{\partial r^\nu} \in C(\bar{A}) \), \( x \in \bar{A} \), \( A \) is a spherical shell:
$$A := B(0, R_2) - B(0, R_1) \subseteq \mathbb{R}^N, \; N \geq 2. \; \text{Furthermore assume that } \frac{\partial^2 f}{\partial r^\beta} = 0, \; \text{for } j = 0, 1, \ldots, n - 1, \; \text{on } \partial B(0, R_1). \; \text{Then}$$

$$\int_A \left| \frac{\partial^\gamma f(x)}{\partial r^\gamma} \right|^\beta \left| \frac{\partial^\nu f(x)}{\partial r^\nu} \right|^{\alpha} dx \leq K \int_A \left| \frac{\partial^\gamma f(x)}{\partial r^\gamma} \right|^{\alpha+\beta} dx. \tag{2.57}$$

Here

$$= \left( \frac{\alpha}{\alpha + \beta} \right)^{\alpha/(\alpha+\beta)} \frac{1}{(\Gamma(\nu - \gamma))^\beta} \int_{R_1}^{R_2} r^{N-1} \left( (P_1(r))^{(\alpha+\beta-1)} dr \right)^{\frac{\beta}{\alpha+\beta}}, \tag{2.58}$$

with

$$P_1(r) := \int_{R_1}^{r} t^{1-N-\beta} (r-t)^{(\nu-\gamma-1)\frac{\beta}{\alpha+\beta}} dr. \tag{2.59}$$

**Proof.** The assumption imply that $f(r\omega) \in C^\alpha([R_1, R_2])$ and $\frac{\partial^\nu f(r\omega)}{\partial r^\nu} \in C([R_1, R_2])$, $\forall \omega \in S^{N-1}$. Hence by Theorem 2.7, $\forall \omega \in S^{N-1}$ we get that

$$\int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial^\gamma f(r\omega)}{\partial r^\gamma} \right|^\beta \left| \frac{\partial^\nu f(r\omega)}{\partial r^\nu} \right|^{\alpha} dr$$

$$\leq K \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial^\nu f(r\omega)}{\partial r^\nu} \right|^{\alpha+\beta} dr. \tag{2.60}$$

Therefore it holds

$$\int_{S^{N-1}} \left( \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial^\gamma f(r\omega)}{\partial r^\gamma} \right|^\beta \left| \frac{\partial^\nu f(r\omega)}{\partial r^\nu} \right|^{\alpha} dr \right) d\omega$$

$$= K \left( \int_{S^{N-1}} \left( \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial^\nu f(r\omega)}{\partial r^\nu} \right|^{\alpha+\beta} dr \right) d\omega \right). \tag{2.61}$$

Using Lemma 2.1 and by (1.3) we derive (2.57).

We need

**Theorem 2.9.** (see Anastassiou and Goldstein [9]) Let $\nu \geq 2, \; \alpha, \beta > 0, \; r > \alpha, \; r > 1; \; p > 0, \; q \geq 0$ be continuous functions on $[a, b]$. Let $f \in C^\nu_{x_0}([a, b])$ with $f^{(i)}(x_0) = 0, \; i = 0, 1, \ldots, n - 1, \; n := [\nu]$. Let $x, x_0 \in [a, b]$ with $x \geq x_0$. Then

$$\int_{x_0}^{x} g(w) |f(w)|^\beta |D_x^\nu f(x)|^\alpha dw \leq K^*(p, q, \nu, \alpha, \beta, r, x, x_0)$$

$$\cdot \left( \int_{x_0}^{x} p(w) |D_x^\nu f(w)|^r dw \right)^{\frac{\alpha+\beta}{\alpha}}. \tag{2.62}$$

Here

$$K^*(p, q, \nu, \alpha, \beta, r, x, x_0) := \left( \frac{\alpha}{\alpha + \beta} \right)^{\alpha/r} \cdot \frac{1}{(\Gamma(\nu))^\beta} \cdot \left( \int_{x_0}^{x} \left( (q(w))^r \cdot (p(w))^{-\alpha} \right)^{\frac{1}{r-\alpha}} \cdot (P_1^*(w))^{\frac{\beta(r-1)}{r-\alpha}} \cdot dw \right)^{\frac{\alpha+\beta}{\alpha}}, \tag{2.63}$$
with
\[ P_1^*(w) := \int_{x_0}^{w} (p(t))^{\frac{1}{p-1}} \cdot (w - t)^{\frac{1}{p-1}} (\nu - (\nu - 1)p/2) - (\nu - 1) - (p-1)/2) dt. \] (2.64)

Based on Theorem 2.9 we give similarly:

**Theorem 2.10.** Let \( \nu \geq 2, \ n := [\nu], \ \alpha, \beta > 0, \ \alpha + \beta > 1. \) Let \( f \in C^n(A) \) and there exists \( \frac{\partial f(x)}{\partial \nu} \in C(A), \ x \in A, \ A \) is a spherical shell: \( A := B(0, R_2) - B(0, R_1) \subseteq \mathbb{R}^N, \ N \geq 2. \) Furthermore assume that \( \frac{\partial f}{\partial \nu} = 0, \ for j = 0, 1, \ldots, n - 1, \ on \ \partial B(0, R_1). \) Then
\[ \int_A |f(x)|^\beta \left| \frac{\partial R_1 f(x)}{\partial \nu} \right|^\alpha dx \leq K^* \int_A \left| \frac{\partial R_1 f(x)}{\partial \nu} \right|^{\alpha + \beta} dx. \] (2.65)

Here
\[ K^* := \left( \frac{\alpha}{\alpha + \beta} \right) \left( \frac{2}{\alpha + \beta} \right) \left( \frac{1}{\Gamma(\nu)} \right)^\beta \left( \int_{R_1}^{R_2} r^{N-1} (P_1^*(r))^{(\alpha + \beta - 1)} dr \right)^\frac{\beta}{\alpha + \beta} , \] (2.66)

with
\[ P_1^*(r) := \int_{R_1}^{r} \left( \frac{2}{\alpha + \beta} \right) (r - t)^{(\alpha - 1)} dt. \] (2.67)

Next we present a set of multivariate fractional Opial type inequalities involving two functions over the shell.

We need

**Theorem 2.11.** (see Anastassiou [4]) Let \( \nu, \gamma_1, \gamma_2 \geq 1, \ such that \ \nu - 1 \geq 1, \ \nu - \gamma_2 \geq 1 \) and \( f_1, f_2 \in C^\nu([a, b]) \) with
\[ f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, \ i = 0, 1, \ldots, n - 1, \ n := [\nu]. \] (2.68)

Here, \( x, x_0 \in [a, b] : x \geq x_0. \) Consider also \( p(t) > 0, \) and \( q(t) \geq 0 \) continuous functions on \( [x_0, b]. \)

Let \( \lambda_\nu > 0 \) and \( \lambda_\alpha, \lambda_\beta \geq 0, \ such that \ \lambda_\nu < p, \ where \ p > 1. \) Set
\[ P_k(w) := \int_{x_0}^{w} (w - t)^{(\nu - 1)(\nu - 1)p/2} dt, \]
\[ k = 1, 2, \ x_0 \leq w \leq b, \] (2.69)
\[ A(w) := \frac{q(w) \cdot (P_1(w))^\lambda_\alpha \cdot (P_2(w))^\lambda_\beta}{(\Gamma(\nu - 1))^\lambda_\alpha \cdot (\Gamma(\nu - 2))^\lambda_\beta}, \] (2.70)
\[ A_0(x) := \left( \int_{x_0}^{x} A(w)^{p/2} dw \right)^{(p-\lambda_\nu)/p}, \] (2.71)

and
\[ \delta_1 := \begin{cases} 2^{1-(\lambda_\alpha + \lambda_\nu)/p}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p, \\ 1, & \text{if } \lambda_\alpha + \lambda_\nu \geq p. \end{cases} \] (2.72)
If $\lambda_\beta = 0$, we obtain that,
\[
\int_{x_0}^{x} q(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \cdot \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\nu} + \left| (D_{x_0}^{\gamma_2} f_1)(w) \right|^{\lambda_\alpha} \cdot \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\nu} \right] \, dw
\leq (A_0(x) \mid_{\lambda_\beta = 0}) \cdot \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu/p} \cdot \delta_1 \cdot \int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^{\nu} f_1)(w) \right|^p + \left| (D_{x_0}^{\nu} f_2)(w) \right|^p \right] \, dw \right]^{(\lambda_\alpha + \lambda_\nu)/p}.
\] (2.73)

Similarly, by (2.73), we derive

**Theorem 2.12.** Let $\nu, \gamma_1, \gamma_2 \geq 1$, such that $\nu - \gamma_1 \geq 1$, $\nu - \gamma_2 \geq 1$, $n := [\nu]$ and $f_1, f_2 \in C^n(A)$ and there exist $\frac{\partial R_1 f_1(x)}{\partial r^\nu}, \frac{\partial R_1 f_2(x)}{\partial r^\nu} \in C(A)$, $A := B(0, R_2) - B(0, R_1) \subseteq \mathbb{R}^N$, $N \geq 2$. Furthermore assume $\frac{\partial R_1 f_i}{\partial r_j} = \frac{\partial R_2 f_i}{\partial r_j} = 0$, for $j = 0, 1, \ldots, n - 1$, on $\partial B(0, R_1)$.

Let $\lambda_\nu > 0$ and $\lambda_\alpha > 0$; $\lambda_\beta \geq 0$, $p := \lambda_\alpha + \lambda_\nu > 1$. Set
\[
P_k(w) := \int_{R_1}^{w} (w - t)^{(\nu - \gamma_1 - 1)p/(p - 1)} t^{\left(\frac{\nu - \gamma_1}{p - 1}\right)} \, dt,
\] (2.74) $k = 1, 2$, $R_1 \leq w \leq R_2$,
\[
A(w) := \frac{w^{(N-1)(1-\frac{1}{p})} (P_1(w))^{\lambda_\alpha} (P_2(w))^{\lambda_\beta}}{(\Gamma(\nu - \gamma_1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2))^{\lambda_\beta}},
\] (2.75)
\[
A_0(R_2) := \left( \int_{R_1}^{R_2} (A(w))^{\frac{1}{\lambda_\alpha}} \, dw \right)^{\frac{1}{\lambda_\beta}}.
\] (2.76)

Take the case of $\lambda_\beta = 0$. Then
\[
\int_A \left[ \left| \frac{\partial R_1 f_1(x)}{\partial r^\nu}(x) \right|^{\lambda_\nu} \cdot \left| \frac{\partial R_1 f_2(x)}{\partial r^\nu}(x) \right|^{\lambda_\nu} \right] \, dx
\leq (A_0(R_2)|_{\lambda_\beta = 0}) \left( \frac{\lambda_\nu}{p} \right)^{\lambda_\nu/p} \int_A \left[ \left| \frac{\partial R_1 f_1(x)}{\partial r^\nu}(x) \right|^p + \left| \frac{\partial R_1 f_2(x)}{\partial r^\nu}(x) \right|^p \right] \, dx.
\] (2.77)

We need

**Theorem 2.13.** (see Anastassiou [4]) All here, as in Theorem 2.11. Denote
\[
\delta_3 := \begin{cases} 
2^{\lambda_3/\lambda_\nu} - 1, & \text{if } \lambda_\beta \geq \lambda_\nu, \\
1, & \text{if } \lambda_\beta \leq \lambda_\nu.
\end{cases}
\]
If \( \lambda_{\alpha} = 0 \), then, it holds
\[
\int_{x_0}^{x} q(w) \left[ \left( (D_{x_0}^{\gamma_2} f_2)(w) \right)^{\lambda_{\beta}} \cdot \left( (D_{x_0}^{\nu} f_1)(w) \right)^{\lambda_{\nu}} 
+ \left( (D_{x_0}^{\gamma_2} f_1)(w) \right)^{\lambda_{\beta}} \cdot \left( (D_{x_0}^{\nu} f_2)(w) \right)^{\lambda_{\nu}} \right] dw
\]
\[
\leq (A_0(x)|_{\lambda_{\alpha}=0}) 2^{p-\lambda_{\nu}/p} \left( \frac{\lambda_{\nu}}{\lambda_{\beta} + \lambda_{\nu}} \right)^{\lambda_{\nu}/p} \delta_{3}^{\lambda_{\nu}/p}.
\]

All \( x_0 \leq x \leq b \).

Similarly, by (2.78), we derive

**Theorem 2.14.** All basic assumptions as in Theorem 2.12. Let \( \lambda_{\nu} > 0, \lambda_{\alpha} = 0, \lambda_{\beta} > 0, p := \lambda_{\nu} + \lambda_{\beta} > 1, P_2 \) defined by (2.74). Now it is
\[
A(w) := \frac{w^{(N-1)(1-\frac{\lambda_{\nu}}{\nu})} (P_2(w))^{\lambda_{\beta}(\frac{\nu-1}{\nu})}}{(\Gamma(\nu-\gamma_2))^{\lambda_{\beta}}},
\]
\[
A_0(R_2) := \left( \int_{R_1}^{R_2} (A(w))^{\frac{\nu}{\lambda_{\nu}}} dw \right)^{\lambda_{\beta}/p}.
\]

Denote
\[
\delta_{3} := \begin{cases} 
2^{\lambda_{\beta}/\lambda_{\nu}} - 1, & \text{if } \lambda_{\beta} \geq \lambda_{\nu}, \\
1, & \text{if } \lambda_{\beta} \leq \lambda_{\nu}.
\end{cases}
\]

Then
\[
\int_{A} \left[ \left( \frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}} \right)^{\lambda_{\beta}} \left( \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right)^{\lambda_{\nu}} + \left( \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right)^{\lambda_{\beta}} \left( \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right)^{\lambda_{\nu}} \right] dx
\]
\[
\leq A_0(R_2) 2^{\lambda_{\beta}/p} \left( \frac{\lambda_{\nu}}{p} \right)^{(\lambda_{\nu}/p)} \delta_{3}^{\lambda_{\nu}/p} \int_{A} \left( \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{p} + \left| \frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\nu}} \right|^{p} \right) dx.
\]

We need

**Theorem 2.15.** (see Anastassiou [4]) All here, as in Theorem 2.11 (\( \lambda_{\alpha}, \lambda_{\beta} \neq 0 \)).

Denote
\[
\tilde{\gamma}_1 := \begin{cases} 
2^{(\lambda_{\alpha} + \lambda_{\beta})/\lambda_{\nu}} - 1, & \text{if } \lambda_{\alpha} + \lambda_{\beta} \geq \lambda_{\nu}, \\
1, & \text{if } \lambda_{\alpha} + \lambda_{\beta} \leq \lambda_{\nu},
\end{cases}
\]
and
\[
\tilde{\gamma}_2 := \begin{cases} 
1, & \text{if } \lambda_{\alpha} + \lambda_{\beta} + \lambda_{\nu} \geq p, \\
2^{1-(\lambda_{\alpha} + \lambda_{\beta} + \lambda_{\nu})/p}, & \text{if } \lambda_{\alpha} + \lambda_{\beta} + \lambda_{\nu} \leq p.
\end{cases}
\]

Then, it holds
\[
\int_{x_0}^{x} q(w) \left[ \left( (D_{x_0}^{\gamma_1} f_1)(w) \right)^{\lambda_{\alpha}} \cdot \left( (D_{x_0}^{\gamma_2} f_2)(w) \right)^{\lambda_{\beta}} \cdot \left( (D_{x_0}^{\nu} f_1)(w) \right)^{\lambda_{\nu}} \right] dw
\]
\[ + \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\nu} \cdot \left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\nu} \cdot \left| (D_{x_0}^{\nu} f_3) (w) \right|^{\lambda_\nu} \right] \, dw \\
\leq A_0(x) \left( \frac{\lambda_\nu}{(\lambda_{\alpha} + \lambda_{\beta})(\lambda_{\alpha} + \lambda_{\beta} + \lambda_\nu)} \right)^{\lambda_\nu/p} \left[ \lambda_{\alpha}^{\lambda_{\nu}/p} \gamma_1^{2(\nu - \lambda_\nu)/(\nu - \gamma_1 \lambda_{\beta})} \lambda_{\nu}/p \right]. \\
\left( \int_{x_0}^{x} p(w) \left( \left| (D_{x_0}^{\nu} f_1) (w) \right|^{\lambda_\nu} + \left| (D_{x_0}^{\nu} f_2) (w) \right|^{\lambda_\nu} \right) \, dw \right)^{(\lambda_{\alpha} + \lambda_{\beta} + \lambda_\nu)/p}, \quad (2.85) \\n\text{all } x_0 \leq x \leq b.
\]

Similarly, by (2.85), we obtain

**Theorem 2.16.** Let all basics as in Theorem 2.12. Here, \( \lambda_\nu, \lambda_{\alpha}, \lambda_{\beta} > 0 \), \( p := \lambda_{\alpha} + \lambda_{\beta} + \lambda_\nu > 1 \). Also \( P_k, k = 1, 2 \) as in (2.74), and \( A(w) \) as in (2.75). Here it is

\[ A_0(R_2) := \left( \int_{R_1}^{R_2} (A(w))^{\lambda_{\alpha} + \lambda_{\beta}} \, dw \right)^{1/p}, \quad (2.86) \]

\[ \gamma_1 := \begin{cases} 
2((\lambda_{\alpha} + \lambda_{\beta})/\lambda_\nu) - 1, & \text{if } \lambda_{\alpha} + \lambda_{\beta} \geq \lambda_\nu, \\
1, & \text{if } \lambda_{\alpha} + \lambda_{\beta} \leq \lambda_\nu. 
\end{cases} \quad (2.87) \]

Then

\[ \int_{\mathbb{R}} \left[ \left| \frac{\partial_{\gamma_1}^2 f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\nu} \cdot \left| \frac{\partial_{\gamma_1}^2 f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\nu} \cdot \left| \frac{\partial_{\gamma_1}^2 f_3(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\nu} \right] \, dx \\
\leq A_0(R_2) \left( \frac{\lambda_\nu}{(\lambda_{\alpha} + \lambda_{\beta})p} \right)^{(\lambda_{\nu}/p)} \left[ \lambda_{\alpha}^{\lambda_{\nu}/p} + 2((\lambda_{\alpha} + \lambda_{\beta})/(\gamma_1 \lambda_{\beta})^{\lambda_{\nu}/p}) \right] \\
\left( \int_{\mathbb{R}} \left( \left| \frac{\partial_{\gamma_1}^2 f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\nu} + \left| \frac{\partial_{\gamma_1}^2 f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\nu} \right) \, dx \right). \quad (2.88) \]

We need

**Theorem 2.17.** (see Anastassiou [4]) Let \( \nu \geq 3 \) and \( \gamma_1 \geq 1 \), such that \( \nu - \gamma_1 \geq 2 \). Let \( f_1, f_2 \in C_{2,x_0}^{\nu}([a, b]) \) with

\[ f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, \quad i = 0, 1, \ldots, n - 1, \]

\( n := [\nu] \). Here \( x, x_0 \in [a, b] : x \geq x_0 \). Consider also, \( p(t) > 0 \), and \( q(t) \geq 0 \) continuous functions on \([x_0, b]\). Let

\[ \lambda_{\alpha} \geq 0, \quad 0 < \lambda_{\alpha + 1} < 1, \]

and \( p > 1 \). Denote

\[ \theta_3 := \begin{cases} 
2\lambda_{\alpha}/(\lambda_{\alpha + 1}) - 1, & \text{if } \lambda_{\alpha} \geq \lambda_{\alpha + 1}, \\
1, & \text{if } \lambda_{\alpha} \leq \lambda_{\alpha + 1}, 
\end{cases} \]

\[ L(x) := \left( 2 \int_{x_0}^{x} (q(w))^{(1/(1-(\lambda_{\alpha + 1}))} \, dw \right)^{(1-\lambda_{\alpha + 1})} \left( \frac{\theta_3 \lambda_{\alpha + 1}}{\lambda_{\alpha} + \lambda_{\alpha + 1}} \right)^{\lambda_{\alpha + 1}} \lambda_{\alpha + 1}, \quad (2.89) \]
Theorem 2.18. Let \( \nu \geq 3, \gamma_1 \geq 1 \), such that \( \nu - \gamma_1 \geq 2 \), \( n := [\nu] \). Let \( f_1, f_2 \in C^n(\bar{A}) \) and there exist \( \frac{\partial^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}}, \frac{\partial^{\gamma_1} f_2(x)}{\partial r^{\gamma_1}} \in C(\bar{A}) \), \( A := B(0, R_2) - B(0, R_1) \subseteq \mathbb{R}^N, N \geq 2 \). Furthermore assume \( \frac{\partial^j f_1}{\partial r^j} = \frac{\partial^j f_2}{\partial r^j} = 0, j = 0, 1, \ldots, n-1 \), on \( \partial B(0, R_1) \).

Let \( \lambda_\alpha > 0, 0 < \lambda_{\alpha+1} < 1 \), such that \( p := \lambda_\alpha + \lambda_{\alpha+1} > 1 \).

Denote

\[
\theta_3 := \left\{ \begin{array}{ll}
2^{(\lambda_\alpha/\lambda_{\alpha+1})} - 1 & \text{if } \lambda_\alpha \geq \lambda_{\alpha+1} \\
1 & \text{if } \lambda_\alpha \leq \lambda_{\alpha+1},
\end{array} \right.
\]

Then, it holds

\[
P_1(x) := \int_{x_0}^x (x - t)^{(\nu-\gamma_1-1)p/(p-1)}(p(t))^{-1/(p-1)} \, dt,
\]

\[
T(x) := L(x) \cdot \left( \frac{P_1(x)}{(p-1)/p} \right)^{(\lambda_\alpha + \lambda_{\alpha+1}) / (\nu - \gamma_1)},
\]

and

\[
\omega_1 := \left\{ \begin{array}{ll}
2^{1-(\lambda_\alpha + \lambda_{\alpha+1})/p} & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \leq p, \\
1 & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \geq p,
\end{array} \right.
\]

\[
\Phi(x) := T(x) \omega_1.
\]

Then, it holds

\[
\int_{x_0}^x q(w) \left[ \left| (D_0^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \cdot \left| (D_0^{\gamma_1} f_2)(w) \right|^{\lambda_{\alpha+1}} + \\
\left| (D_0^{\gamma_1} f_2)(w) \right|^{\lambda_\alpha} \cdot \left| (D_0^{\gamma_1} f_1)(w) \right|^{\lambda_{\alpha+1}} \right] \, dw
\]

\[
\leq \Phi(x) \left[ \int_{x_0}^x p(w) \cdot \left| (D_0^{\gamma_1} f_1)(w) \right|^p + \left| (D_0^{\gamma_1} f_2)(w) \right|^p \, dw \right]^{(\lambda_\alpha + \lambda_{\alpha+1})/p},
\]

all \( x_0 \leq x \leq b \).

Similarly, by (2.94), we obtain

\[
T(x) := \left( \frac{P_1(x)}{(p-1)/p} \right)^{(\lambda_\alpha + \lambda_{\alpha+1}) / (\nu - \gamma_1)},
\]

\[
\omega_1 := \left\{ \begin{array}{ll}
2^{1-(\lambda_\alpha + \lambda_{\alpha+1})/p} & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \leq p, \\
1 & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \geq p,
\end{array} \right.
\]

\[
\Phi(x) := T(x) \omega_1.
\]

Then, it holds

\[
\int_{x_0}^x q(w) \left[ \left| (D_0^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \cdot \left| (D_0^{\gamma_1} f_2)(w) \right|^{\lambda_{\alpha+1}} + \\
\left| (D_0^{\gamma_1} f_2)(w) \right|^{\lambda_\alpha} \cdot \left| (D_0^{\gamma_1} f_1)(w) \right|^{\lambda_{\alpha+1}} \right] \, dw
\]

\[
\leq \Phi(x) \left[ \int_{x_0}^x p(w) \cdot \left| (D_0^{\gamma_1} f_1)(w) \right|^p + \left| (D_0^{\gamma_1} f_2)(w) \right|^p \, dw \right]^{(\lambda_\alpha + \lambda_{\alpha+1})/p},
\]

all \( x_0 \leq x \leq b \).

Similarly, by (2.94), we obtain
\[ \leq \Phi(R_2) \int_A \left( \left| \frac{\partial^\nu_{R_1} f_1(x)}{\partial r^\nu} \right|^p + \left| \frac{\partial^\nu_{R_1} f_2(x)}{\partial r^\nu} \right|^p \right) dx. \] (2.99)

We need

**Theorem 2.19.** (see Anastassiou [4]) *All here, as in Theorem 2.11. Consider the special case \( \lambda_\beta = \lambda_\alpha + \lambda_\nu. \) Denote

\[ T(x) := A_0(x) \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\omega/p} 2^{(\nu - 2\lambda_\alpha - 3\lambda_\omega)/p}. \] (2.100)

Then, it holds

\[ \int_{x_0}^x q(w) \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\omega} \left| (D_{x_0}^{\gamma_2} f_2)(w) \right|^{\lambda_\alpha + \lambda_\nu} \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\nu} \right. \]
\[ + \left. \left| (D_{x_0}^{\nu} f_1)(w) \right|^{\lambda_\alpha + \lambda_\nu} \left| (D_{x_0}^{\nu} f_2)(w) \right|^{\lambda_\omega} \left| (D_{x_0}^{\nu} f_2)(w) \right|^{\lambda_\nu} \right] dw \]
\[ \leq T(x) \left( \int_{x_0}^x p(w) \left( \left| (D_{x_0}^{\nu} f_1)(w) \right|^p + \left| (D_{x_0}^{\nu} f_2)(w) \right|^p \right) dw \right) 2^{(\lambda_\alpha + \lambda_\nu)/p}, \] (2.101)

all \( x_0 \leq x \leq b. \)

Similarly, by (2.101), we get

**Theorem 2.20.** *Here all as in Theorem 2.12. Consider the case \( \lambda_\beta = \lambda_\alpha + \lambda_\nu; \lambda_\alpha \geq 0, \lambda_\nu > 0, \lambda_\beta > \frac{1}{2}, p := 2\lambda_\beta. \) Here \( P_k, k = 1, 2, \) as in (2.74) and \( A(w) \) as in (2.75). Set

\[ A_0(R_2) := \left( \int_{R_1}^{R_2} (A(w))^{p/(2\lambda_\alpha + \lambda_\nu)} \right)^{\frac{(2\lambda_\alpha + \lambda_\omega)}{p}}. \] (2.102)

Also put

\[ \bar{T}(R_2) := A_0(R_2) \left( \frac{\lambda_\nu}{\lambda_\beta} \right)^{\lambda_\omega} 2^{\left(-\frac{\lambda_\omega}{p}\right)}. \] (2.103)

Then

\[ \int_A \left[ \left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\omega} \left| \frac{\partial_{R_1}^{\gamma_2} f_2(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha + \lambda_\nu} \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right. \]
\[ + \left. \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\alpha + \lambda_\nu} \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right|^{\lambda_\omega} \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] dx \]
\[ \leq \bar{T}(R_2) \int_A \left( \left| \frac{\partial_{R_1}^{\nu} f_1(x)}{\partial r^{\nu}} \right|^p + \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right|^p \right) dx. \] (2.104)

We need

**Theorem 2.21.** (see Anastassiou [4]) *Let \( \nu, \gamma_1, \gamma_2 \geq 1, \) such that \( \nu - \gamma_1 \geq 1, \nu - \gamma_2 \geq 1 \) and \( f_1, f_2 \in C_{x_0}^\nu([a, b]) \) with \( f_1^{(i)}(x_0) = f_2^{(i)}(x_0) = 0, i = 0, 1, \ldots, n - 1, n := \)
Then, it holds
\[
\int_{x_0}^{x} q(w) \left[ \left| (D_{x_0}^\gamma f_1) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^\gamma f_2) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^\nu f_1) (w) \right|^{\lambda_\nu} + \left| (D_{x_0}^\nu f_1) (w) \right|^{\lambda_\beta} \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_\nu} \right] dw 
\leq \frac{\rho(x)}{2} \left[ \left\| D_{x_0}^\nu f_1 \right\|_{\infty}^{2(\lambda_\alpha + \lambda_\nu)} + \left\| D_{x_0}^\nu f_1 \right\|_{\infty}^{2\lambda_\beta} + \left\| D_{x_0}^\nu f_2 \right\|_{\infty}^{2\lambda_\beta} + \left\| D_{x_0}^\nu f_2 \right\|_{\infty}^{2(\lambda_\alpha + \lambda_\nu)} \right] ,
\] (2.106)
all \( x_0 \leq x \leq b \).

Similarly, by (2.106), we get

**Theorem 2.22.** Same basic assumptions as in Theorem 2.12. Let \( \lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0 \). Set
\[
\rho(R_2) := \frac{R_2^{N-1} (R_2 - R_1)^{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\beta + 1)} (\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}}{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\beta + 1) (\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha} (\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}} .
\] (2.107)

Then
\[
\int_A \left[ \left| \frac{\partial^\gamma_{R_1} f_1 (x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial^\gamma_{R_1} f_2 (x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial^\nu_{R_1} f_1 (x)}{\partial r^{\nu}} \right|^{\lambda_\nu} + \left| \frac{\partial^\gamma_{R_1} f_1 (x)}{\partial r^{\gamma_1}} \right|^{\lambda_\beta} \left| \frac{\partial^\nu_{R_1} f_2 (x)}{\partial r^{\nu}} \right|^{\lambda_\nu} \right] dx 
\leq \rho(R_2) \frac{\pi^{N/2}}{\Gamma(N/2)} \left[ \left\| \frac{\partial^\nu_{R_1} f_1}{\partial r^{\nu}} \right\|_{\infty}^{2(\lambda_\alpha + \lambda_\nu)} + \left\| \frac{\partial^\nu_{R_1} f_1}{\partial r^{\nu}} \right\|_{\infty}^{2\lambda_\beta} + \left\| \frac{\partial^\nu_{R_1} f_2}{\partial r^{\nu}} \right\|_{\infty}^{2(\lambda_\alpha + \lambda_\nu)} \right] .
\] (2.108)

We need

**Theorem 2.23.** (see Anastassiou [4]) (Assume, as in Theorem 2.21, \( \lambda_\beta = 0 \).) It holds
\[
\int_{x_0}^{x} p(w) \left[ \left| (D_{x_0}^\gamma f_1) (w) \right|^{\lambda_\alpha} \cdot \left| (D_{x_0}^\nu f_1) (w) \right|^{\lambda_\nu} + \left| (D_{x_0}^\gamma f_2) (w) \right|^{\lambda_\alpha} \cdot \left| (D_{x_0}^\nu f_2) (w) \right|^{\lambda_\nu} \right] dw 
\leq \left( \frac{(x - x_0)^{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + 1)} \left\| p(x) \right\|_{\infty}}{(\nu \lambda_\alpha - \gamma_1 \lambda_\alpha + 1) (\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \right) \cdot \left[ \left\| D_{x_0}^\nu f_1 \right\|_{\infty}^{\lambda_\alpha + \lambda_\nu} + \left\| D_{x_0}^\nu f_2 \right\|_{\infty}^{\lambda_\alpha + \lambda_\nu} \right] ,
\] (2.109)
\[ \int_{A} \left[ \left| \frac{\partial^{n}_{R_{1}} f_{1}(x)}{\partial r^{\gamma_{1}}} \right|^{\lambda_{a}} \left| \frac{\partial^{n}_{R_{1}} f_{2}(x)}{\partial r^{\gamma_{2}}} \right|^{\lambda_{b}} + \left| \frac{\partial^{n}_{R_{1}} f_{1}(x)}{\partial r^{\gamma_{1}}} \right|^{\lambda_{a}} \left| \frac{\partial^{n}_{R_{1}} f_{2}(x)}{\partial r^{\gamma_{2}}} \right|^{\lambda_{b}} \right] \, dx \]

\[ \leq \frac{2\pi^{N/2}}{\Gamma(N/2)} \frac{R_{2}^{N-1}(R_{2} - R_{1})^{(\nu \lambda_{a} - \gamma_{1} \lambda_{a} + 1)}(\Gamma(\nu - \gamma_{1} + 1))^{\lambda_{a}}}{\left( N = (2\nu \lambda_{a} - \gamma_{1} \lambda_{a} + \nu \lambda_{b} - \gamma_{2} \lambda_{a} - \gamma_{2} \lambda_{b} + 1)(\Gamma(\nu - \gamma_{1} + 1))^\lambda \right)} \cdot \left( \left| \frac{\partial^{n}_{R_{1}} f_{2}}{\partial r^{\gamma_{1}}} \right|^{\lambda_{a} + \lambda_{b}} \left| \frac{\partial^{n}_{R_{1}} f_{1}}{\partial r^{\gamma_{2}}} \right|^{\lambda_{a} + \lambda_{b}} \right) \right]. \]  

(2.110)

We need

**Theorem 2.25.** (see Anastassiou [4]) (In relationship to Theorem 2.21, \( \lambda_{\beta} = \lambda_{a} + \lambda_{b} \).) It holds

\[ \int \limits_{x_{0}}^{x} p(w) \left[ \left| (D_{x_{0}}^{n} f_{1}) (w) \right|^{\lambda_{a}} \left| (D_{x_{0}}^{n} f_{2}) (w) \right|^{\lambda_{b}} \right] \, dw \]

\[ \leq \frac{1}{\Gamma(\nu - \gamma_{2} + 1)^{\lambda_{a} + \lambda_{b}}} \left( \left| D_{x_{0}}^{n} f_{1} \left( D_{x_{0}}^{n} f_{2} \right) \right|^{\lambda_{a} + \lambda_{b}} \right) \]  

all \( x_{0} \leq x \leq b \).

Similarly, by (2.111), we derive

**Theorem 2.26.** All as in Theorem 2.22. Assume \( \lambda_{\beta} = \lambda_{a} + \lambda_{b} \). Then

\[ \int_{A} \left[ \left| \frac{\partial^{n}_{R_{1}} f_{1}(x)}{\partial r^{\gamma_{1}}} \right|^{\lambda_{a}} \left| \frac{\partial^{n}_{R_{1}} f_{2}(x)}{\partial r^{\gamma_{2}}} \right|^{\lambda_{b}} + \left| \frac{\partial^{n}_{R_{1}} f_{1}(x)}{\partial r^{\gamma_{1}}} \right|^{\lambda_{a}} \left| \frac{\partial^{n}_{R_{1}} f_{2}(x)}{\partial r^{\gamma_{2}}} \right|^{\lambda_{b}} \right] \, dx \]

\[ \leq \left( \frac{2\pi^{N/2}}{\Gamma(N/2)} \right) \frac{R_{2}^{N-1}(R_{2} - R_{1})^{(2\nu \lambda_{a} - \gamma_{1} \lambda_{a} + \nu \lambda_{b} - \gamma_{2} \lambda_{a} - \gamma_{2} \lambda_{b} + 1)}(\Gamma(\nu - \gamma_{1} + 1))^\lambda}{\left( (2\nu \lambda_{a} - \gamma_{1} \lambda_{a} + \nu \lambda_{b} - \gamma_{2} \lambda_{a} - \gamma_{2} \lambda_{b} + 1)(\Gamma(\nu - \gamma_{1} + 1))^\lambda \right)} \cdot \left( \left| \frac{\partial^{n}_{R_{1}} f_{2}}{\partial r^{\gamma_{1}}} \right|^{\lambda_{a} + \lambda_{b}} \left| \frac{\partial^{n}_{R_{1}} f_{1}}{\partial r^{\gamma_{2}}} \right|^{\lambda_{a} + \lambda_{b}} \right). \]  

(2.112)

We need
Theorem 2.27. (see Anastassiou [4]) (In relationship to Theorem 2.21, \( \lambda_\nu = 0, \lambda_\alpha = \lambda_\beta \).) It holds
\[
\int_{x_0}^b p(w) \left[ |(D_{x_0}^{\gamma_1} f_1)(w)|^{\lambda_\alpha} \cdot |(D_{x_0}^{\gamma_2} f_2)(w)|^{\lambda_\beta} + \\
| (D_{x_0}^{\gamma_2} f_1)(w) |^{\lambda_\alpha} \cdot | (D_{x_0}^{\gamma_1} f_2)(w) |^{\lambda_\beta} \right] dw \\
\leq \rho^*(x) \left[ \| D_{x_0}^{\nu} f_1 \|^2_{\infty} + \| D_{x_0}^{\nu} f_2 \|^2_{\infty} \right],
\]
(2.113)
all \( x_0 \leq x \leq b \).

Here
\[
\rho^*(x) := \left( \frac{(x - x_0)^{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha - \gamma_2\lambda_\alpha + 1)} \cdot \| p(x) \|_{\infty}}{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha - \gamma_2\lambda_\alpha + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\alpha}} \right).
\]

(2.114)
We get, by (2.113), the result.

Theorem 2.28. All as in Theorem 2.22. Assume \( \lambda_\nu = 0, \lambda_\alpha = \lambda_\beta \). Then
\[
\int_A \left[ \left| \frac{\partial_{R_1}^\nu f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1} f_2(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} + \left| \frac{\partial_{R_1} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^\nu f_2(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\beta} \right] dx \\
\leq \left( \frac{2\pi^{N/2}}{\Gamma(N/2)} \right) \rho^*(R_2) \left[ \left\| \frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}} \right\|_{\infty}^{2\lambda_\alpha} + \left\| \frac{\partial_{R_1} f_2}{\partial r^{\nu}} \right\|_{\infty}^{2\lambda_\beta} \right],
\]
(2.115)
where
\[
\rho^*(R_2) := \left( \frac{R_2^{N-1}(R_2 - R_1)^{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha - \gamma_2\lambda_\alpha + 1)}}{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha - \gamma_2\lambda_\alpha + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\alpha}} \right).
\]
(2.116)
We need

Theorem 2.29. (see Anastassiou [4]) (In relationship to Theorem 2.21, \( \lambda_\alpha = 0, \lambda_\beta = \lambda_\nu \).) It holds
\[
\int_{x_0}^b p(w) \left[ |(D_{x_0}^{\gamma_2} f_2)(w)|^{\lambda_\beta} \cdot |(D_{x_0}^{\nu} f_1)(w)|^{\lambda_\beta} + \\
| (D_{x_0}^{\nu} f_1)(w) |^{\lambda_\beta} \cdot | (D_{x_0}^{\gamma_2} f_2)(w) |^{\lambda_\beta} \right] dw \\
\leq \left( \frac{(x - x_0)^{(\nu\lambda_\beta - \gamma_2\lambda_\beta + 1)} \cdot \| p(x) \|_{\infty}}{\nu\lambda_\beta - \gamma_2\lambda_\beta + 1)(\Gamma(\nu - \gamma_2 + 1))^{\lambda_\beta}} \right) \cdot \left[ \left\| \frac{\partial_{R_1}^{\nu} f_1}{\partial r^{\nu}} \right\|_{\infty}^{2\lambda_\beta} + \left\| \frac{D_{x_0}^{\nu} f_2}{\partial r^{\nu}} \right\|_{\infty}^{2\lambda_\beta} \right],
\]
(2.117)
all \( x_0 \leq x \leq b \).

We get, by (2.113), the next result.

Theorem 2.30. All as in Theorem 2.22. Assume \( \lambda_\alpha = 0, \lambda_\beta = \lambda_\nu \). Then
\[
\int_A \left[ \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1} f_1(x)}{\partial r^{\nu}} \right|^{\lambda_\beta} + \left| \frac{\partial_{R_1} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} \left| \frac{\partial_{R_1}^{\nu} f_2(x)}{\partial r^{\nu}} \right|^{\lambda_\beta} \right] dx
\]
\[ \left( \frac{2\pi^{N/2}}{\Gamma(N/2)} \right) \left( \frac{R_2^{N-1} (R_2 - R_1)^{(\nu\lambda_\beta - 2\lambda_\beta + 1)}}{(\nu\lambda_\beta - 2\lambda_\beta + 1)(\Gamma(\nu - 2\lambda_\beta + 1))^{\lambda_\beta}} \right) \left[ \left\| \frac{\partial^{\nu}_{\tau} f_1}{\partial \tau^\nu} \right\|_\infty^{2\lambda_\beta} \right] \]

\[ + \left\| \frac{\partial^{\nu}_{\tau} f_2}{\partial \tau^\nu} \right\|_\infty^{2\lambda_\beta} \].

(2.118)

We make

**Assumption 2.1.** Let \( \nu \geq 1, n := [\nu] \), \( f_j \in C^n(\overline{A}), j = 1, \ldots, M \in \mathbb{N} \), and there exist \( \frac{\partial f_j}{\partial \tau^\nu} \in C(\overline{A}), A := B(0, R_2) - B(0, R_1) \subseteq \mathbb{R}^N, N \geq 2 \). Furthermore assume that \( \frac{\partial f_j}{\partial \tau^\nu} = 0, i = 0, 1, \ldots, n-1 \), on \( \partial B(0, R_1) \), for all \( j = 1, \ldots, M \).

Next we present a set of multivariate fractional Opial type inequalities involving several functions over the shell.

We need

**Theorem 2.31.** (see Anastassiou [3]) Let \( \nu, \gamma_1, \gamma_2 \geq 1 \), such that \( \nu - \gamma_1 \geq 1, \nu - \gamma_2 \geq 1 \) and \( f_j \in C^{\nu}_{x_0}([a, b]) \) with \( f_j^{(i)}(x_0) = 0, i = 0, 1, \ldots, n-1, n := [\nu], j = 1, \ldots, M \in \mathbb{N} \). Here, \( x, x_0 \in [a, b] : x \geq x_0 \). Consider also \( p(t) > 0 \), and \( q(t) \geq 0 \) continuous functions on \( [x_0, b] \). Let \( \lambda_\nu > 0 \) and \( \lambda_\alpha, \lambda_\beta \geq 0 \) such that \( \lambda_\nu < p \), where \( p > 1 \).

Set

\[ P_k(w) := \int_{x_0}^{\infty} (w - t)^{(\nu - \gamma_k - 1)p} (p(t))^{-\frac{1}{p-1}} dt, \quad k = 1, 2, x_0 \leq w \leq b; \]

\[ A(w) := \frac{q(w) \cdot (P_1(w))^{\lambda_\alpha(\frac{\nu-1}{p})} \cdot (P_2(w))^{\lambda_\beta(\frac{\nu-1}{p})} (p(w))^{-\frac{\lambda_\nu}{p}}}{(\Gamma(\nu - \gamma_1))^{\lambda_\alpha} \cdot (\Gamma(\nu - \gamma_2))^{\lambda_\beta}}; \]

\[ A_0(x) := \left( \int_{x_0}^{\infty} A(w)^{\frac{\nu}{p - \lambda_\nu}} dw \right)^{\frac{\lambda_\nu}{p}}. \]

**Call**

\[ \varphi_1(x) := \left( A_0(x) \right)_{\lambda_\beta = 0} \cdot \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\frac{\lambda_\nu}{p}}, \]

\[ \delta_1^* := \begin{cases} M^{1 - \frac{\lambda_\alpha + \lambda_\nu}{p}}, & \text{if } \lambda_\alpha + \lambda_\nu \leq p, \\ 2^{\frac{\lambda_\alpha + \lambda_\nu}{p}} - 1, & \text{if } \lambda_\alpha + \lambda_\nu \geq p. \end{cases} \]

If \( \lambda_\beta = 0 \), we obtain that

\[ \int_{x_0}^{x} q(w) \left( \sum_{j=1}^{M} \left| (D_{x_0}^{\gamma_1} f_j)(w) \right|^{\lambda_\alpha} \cdot \left| (D_{x_0}^{\nu} f_j)(w) \right|^{\lambda_\beta} \right) dw \]

\[ \leq \delta_1^* \cdot \varphi_1(x) \cdot \left[ \int_{x_0}^{x} p(w) \left( \sum_{j=1}^{M} \left| (D_{x_0}^{\nu} f_j)(w) \right|^p \right) dw \right]^{\left( \frac{\lambda_\alpha + \lambda_\nu}{p} \right)} \]

all \( x_0 \leq x \leq b. \)

Similarly, by (2.124), we derive
Theorem 2.32. Let $f_{j}, j = 1, \ldots, M,$ as in Assumption 2.1. Let $\gamma_{1}, \gamma_{2} \geq 1,$ such that $\nu - \gamma_{1} \geq 1,$ $\nu - \gamma_{2} \geq 1$ Let $\lambda_{\nu} > 0,$ and $\lambda_{\beta} > 0;$ $\lambda_{\beta} \geq 0,$ $p := \lambda_{\alpha} + \lambda_{\nu} > 1.$ Set
\[
P_{k}(w) := \int_{R_{1}}^{w} (w-t)^{(\nu-\gamma_{k}-1)\frac{p}{p-1}} t^{(1-\frac{p}{p-1})} dt,
\]
for $k = 1, 2, R_{1} \leq w \leq R_{2},$
\[
A(w) := \frac{w^{(N-1)(1-\frac{p}{p-1})}(P_{1}(w))^\lambda_{\alpha}(P_{2}(w))^\lambda_{\beta}}{\Gamma(\nu-\gamma_{1})^\lambda_{\alpha}(\Gamma(\nu-\gamma_{2})^\lambda_{\beta},
\]
\[
A_{0}(R_{2}) := \left(\int_{R_{1}}^{R_{2}} (A(w))^{p/\lambda_{\nu}} dw \right)^{\lambda_{\nu}/p}.
\]
Take the case of $\lambda_{\beta} = 0.$ Then
\[
\sum_{j=1}^{M} \int_{A} \left| \frac{\partial_{R_{1}}^{\gamma_{1}} f_{j}(x)}{\partial r^{\gamma_{1}}} \right|^\lambda_{\nu} \left| \frac{\partial_{R_{1}}^{\gamma_{2}} f_{j}(x)}{\partial r^{\gamma_{2}}} \right|^\lambda_{\nu} dx
\]
\[
\leq (A_{0}(R_{2})|_{\lambda_{\beta}=0}) \left(\frac{\lambda_{\nu}}{p}\right)^{\lambda_{\nu} / p} \left[ M \sum_{j=1}^{M} \left( \int_{A} \left| \frac{\partial_{R_{1}}^{\gamma_{1}} f_{j}(x)}{\partial r^{\gamma_{2}}} \right|^p dx \right) \right].
\]
We need

Theorem 2.33. (see Anastassiou [3]) All here as in Theorem 2.31. Denote
\[
\delta_{\beta} := \begin{cases} 2\frac{\lambda_{\beta}}{\lambda_{\nu}} - 1, & \text{if } \lambda_{\beta} \geq \lambda_{\nu}, \\
1, & \text{if } \lambda_{\beta} \leq \lambda_{\nu},
\end{cases}
\]
\[
\varepsilon_{2} := \begin{cases} 1, & \text{if } \lambda_{\nu} + \lambda_{\beta} \geq p, \\
M^{1-\frac{\lambda_{\nu}+\lambda_{\beta}}{p}}, & \text{if } \lambda_{\nu} + \lambda_{\beta} \leq p,
\end{cases}
\]
and
\[
\varphi_{2}(x) := (A_{0}(x)|_{\lambda_{\alpha}=0}) 2^{\frac{\lambda_{\nu}}{p}} \left( \frac{\lambda_{\nu}}{\lambda_{\beta} + \lambda_{\nu}} \right)^{\lambda_{\nu}/p} \delta_{\beta}^{\lambda_{\nu}/p}.
\]
If $\lambda_{\alpha} = 0,$ then is holds
\[
\int_{x_{0}}^{x} q(w) \left\{ \sum_{j=1}^{M} \left[ (D_{x_{0}}^{\gamma_{2}} f_{j+1})(w) \right]^\lambda_{\beta} \left| (D_{x_{0}}^{\gamma_{2}} f_{j})(w) \right|^\lambda_{\nu} + \left| (D_{x_{0}}^{\gamma_{2}} f_{j})(w) \right|^{\lambda_{\nu}} \right\}
\]
\[
+ \left[ (D_{x_{0}}^{\gamma_{2}} f_{j})(w) \right] \left| (D_{x_{0}}^{\gamma_{2}} f_{j+1})(w) \right|^{\lambda_{\nu}} \right\}
\]
\[
+ \left[ (D_{x_{0}}^{\gamma_{2}} f_{M})(w) \right] \left| (D_{x_{0}}^{\gamma_{2}} f_{j+1})(w) \right|^{\lambda_{\nu}} \right\}
\]
\[
+ \left[ (D_{x_{0}}^{\gamma_{2}} f_{j+1})(w) \right] \left| (D_{x_{0}}^{\gamma_{2}} f_{j+1})(w) \right|^{\lambda_{\nu}} \right\}
\]
\[
\leq 2^{\left(\frac{\lambda_{\nu}+\lambda_{\beta}}{p}\right)} \varepsilon_{2} \varphi_{2}(x) \cdot \left\{ \int_{x_{0}}^{x} p(w) \cdot \left[ \sum_{j=1}^{M} \left| (D_{x_{0}}^{\gamma_{2}} f_{j})(w) \right|^{p} \right] dw \right\}^{\left(\frac{\lambda_{\nu}+\lambda_{\beta}}{p}\right)},
\]
for $x \geq x_{0}.$
Similarly, by (2.132), we obtain
Theorem 2.34. All basic assumptions as in Theorem 2.32. Let \( \lambda_\nu > 0, \lambda_\alpha = 0; \lambda_\beta > 0, \ p := \lambda_\nu + \lambda_\beta > 1, \ P_2 \) defined by (2.125).

Now it is

\[
A(w) := \frac{w^{(N-1)(1-\frac{\lambda_\nu}{p})(P_2(w))^{\lambda_\beta \frac{p-1}{p}}}}{(\Gamma(\nu - \gamma_2))^{\lambda_\beta}},
\]

(2.133)

\[
A_0(R_2) := \left( \int_{R_1}^{R_2} (A(w))^{p/\lambda_\beta} \, dw \right)^{\lambda_\beta/p}.
\]

(2.134)

Denote

\[
\delta_3 := \begin{cases} 
2^{\lambda_\beta/\lambda_\nu} - 1, & \text{if } \lambda_\beta \geq \lambda_\nu, \\
1, & \text{if } \lambda_\beta < \lambda_\nu.
\end{cases}
\]

(2.135)

Call

\[
\varphi_2(R_2) := A_0(R_2) 2^{\lambda_\beta/p} \left( \frac{\lambda_\nu}{p} \right)^{\lambda_\nu/p} \delta_3^{\lambda_\nu/p}.
\]

(2.136)

Then

\[
\int_{A} \left\{ \sum_{j=1}^{M-1} \left[ \left| \frac{\partial_{R_2} f_{j+1}(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} | \frac{\partial_{R_1} f_{j}(x)}{\partial r^{\nu}} |^{\lambda_\nu} \\
+ \left| \frac{\partial_{R_2} f_{j}(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} | \frac{\partial_{R_1} f_{j+1}(x)}{\partial r^{\nu}} |^{\lambda_\nu} \right] \\
+ \left| \frac{\partial_{R_2} f_M(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} | \frac{\partial_{R_1} f_1(x)}{\partial r^{\nu}} |^{\lambda_\nu} \\
+ \left| \frac{\partial_{R_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\beta} | \frac{\partial_{R_1} f_{M}(x)}{\partial r^{\nu}} |^{\lambda_\nu} \right] \right\} \, dx
\]

\[
\leq 2 \varphi_2(R_2) \left[ \sum_{j=1}^{M} \left( \int_{A} | \frac{\partial_{R_1} f_{j}(x)}{\partial r^{\nu}} |^{p} \, dx \right) \right]^{\lambda_\nu/p}.
\]

(2.137)

We need

Theorem 2.35. (see Anastassiou [3]) All here as in Theorem 2.31 \((\lambda_\alpha, \lambda_\beta \neq 0)\).

\[
\gamma_1 := \begin{cases} 
2^{\frac{(\lambda_\alpha + \lambda_\beta)}{\lambda_\nu}} - 1, & \text{if } \lambda_\alpha + \lambda_\beta \geq \lambda_\nu, \\
1, & \text{if } \lambda_\alpha + \lambda_\beta < \lambda_\nu.
\end{cases}
\]

(2.138)

and

\[
\gamma_2 := \begin{cases} 
1, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \geq p, \\
2^{1 - \frac{(\lambda_\alpha + \lambda_\beta + \lambda_\nu - p)}{p}}, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu < p.
\end{cases}
\]

(2.139)

Set

\[
\varphi_3(x) := A_0(x) \cdot \left( \frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)(\lambda_\alpha + \lambda_\beta + \lambda_\nu)} \right)^{\lambda_\nu/p}.
\]

(2.140)
and

\[ \varepsilon_3 := \begin{cases} 1, & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \geq p, \\ M^{-1} \left( \frac{\lambda_\alpha + \lambda_\beta + \lambda_\nu}{p} \right), & \text{if } \lambda_\alpha + \lambda_\beta + \lambda_\nu \leq p, \end{cases} \]

(2.141)

Then it holds

\[
\int_{x_0}^{x} q(w) \left[ \sum_{j=1}^{M-1} \left[ \left( D_{x_0}^{\gamma_1} f_j (w) \right)^{\lambda_\alpha} \right] \left( D_{x_0}^{\gamma_2} f_{j+1} (w) \right)^{\lambda_\beta} \left( D_{x_0}^{\gamma_3} f_j (w) \right)^{\lambda_\nu} \\
+ \left( D_{x_0}^{\gamma_3} f_j (w) \right)^{\lambda_\beta} \left( D_{x_0}^{\gamma_2} f_{j+1} (w) \right)^{\lambda_\alpha} \left( D_{x_0}^{\gamma_1} f_j (w) \right)^{\lambda_\nu} \\
+ \left( D_{x_0}^{\gamma_1} f_1 (w) \right)^{\lambda_\alpha} \left( D_{x_0}^{\gamma_2} f_M (w) \right)^{\lambda_\beta} \left( D_{x_0}^{\gamma_3} f_1 (w) \right)^{\lambda_\nu} \\
+ \left( D_{x_0}^{\gamma_3} f_1 (w) \right)^{\lambda_\beta} \left( D_{x_0}^{\gamma_1} f_M (w) \right)^{\lambda_\alpha} \left( D_{x_0}^{\gamma_2} f_1 (w) \right)^{\lambda_\nu} \right] \, dw \\
\leq 2 \left( \frac{\lambda_\alpha + \lambda_\beta + \lambda_\nu}{p} \right)^{\frac{\lambda_\alpha + \lambda_\beta + \lambda_\nu}{p}} \varepsilon_3 \varphi_3(x) \cdot \left\{ \int_{x_0}^{x} p(w) \left[ \sum_{j=1}^{M} \left( D_{x_0}^{\gamma} f_j (w) \right)^{p} \right] \, dw \right\}^{\frac{\lambda_\alpha + \lambda_\beta + \lambda_\nu}{p}}, \]

(2.142)

all \( x_0 \leq x \leq b \).

Similarly, by (2.142), we obtain

**Theorem 2.36.** All basic assumptions as in Theorem 2.32. Here \( \lambda_\nu, \lambda_\alpha, \lambda_\beta > 0, p := \lambda_\alpha + \lambda_\beta + \lambda_\nu > 1, P_k \) as in (2.125). \( A \) as in (2.126). Here

\[ A_0(R_2) := \left( \int_{R_1}^{R_2} \left( A(w) \right)^{p/(\lambda_\alpha + \lambda_\beta)} \right)^{\frac{\lambda_\alpha + \lambda_\beta}{p}}, \]

(2.143)

\[ \tilde{\gamma}_1 := \begin{cases} 2 \left( \frac{\lambda_\alpha + \lambda_\beta}{\lambda_\nu} \right) - 1, & \text{if } \lambda_\alpha + \lambda_\beta \geq \lambda_\nu, \\
1, & \text{if } \lambda_\alpha + \lambda_\beta \leq \lambda_\nu, \end{cases} \]

(2.144)

Put

\[ \varphi_3(R_2) := A_0(R_2) \left( \frac{\lambda_\nu}{(\lambda_\alpha + \lambda_\beta)p} \right)^{(\lambda_\nu/p)} \left[ \lambda_\alpha^{(\lambda_\nu/p)} + 2 \left( \frac{\lambda_\alpha + \lambda_\beta}{p} \right) \left( \tilde{\gamma}_1 \lambda_\beta \right)^{\lambda_\nu} \right]^{(\lambda_\nu/p)} \]

(2.145)

Then

\[
\int_{A} \left[ \sum_{j=1}^{M-1} \left[ \frac{\partial_1^{\gamma_1} f_j (x)}{\partial x} \right]^{\lambda_\alpha} \left[ \frac{\partial_2^{\gamma_2} f_{j+1} (x)}{\partial x} \right]^{\lambda_\beta} \left[ \frac{\partial_3^{\gamma_3} f_j (x)}{\partial x} \right]^{\lambda_\nu} + \\
\left[ \frac{\partial_1^{\gamma_1} f_1 (x)}{\partial x} \right]^{\lambda_\alpha} \left[ \frac{\partial_2^{\gamma_2} f_M (x)}{\partial x} \right]^{\lambda_\beta} \left[ \frac{\partial_3^{\gamma_3} f_1 (x)}{\partial x} \right]^{\lambda_\nu} \right] \, dx \\
\leq 2 \varphi_3(R_2) \sum_{j=1}^{M} \left( \int_{A} \left| \frac{\partial_1^{\gamma_1} f_j (x)}{\partial x} \right|^p \, dx \right) \]

(2.146)
We need

**Theorem 2.37.** (see Anastassiou [3]) Let \( \nu \geq 3 \), and \( \gamma_1 \geq 1 \), such that \( \nu - \gamma_1 \geq 2 \).

Let \( f_j \in C_x^n([a, b]) \) with \( f_j^{(i)}(x_0) = 0 \), \( i = 0, 1, \ldots, n - 1 \), \( n := [\nu] \), \( j = 1, \ldots, M \in \mathbb{N} \).

Here, \( x, x_0 \in [a, b] : x \geq x_0 \). Consider also \( p(t) > 0 \), and \( q(t) \geq 0 \) continuous functions on \([x_0, b]\). Let \( \lambda_\alpha \geq 0 \), \( 0 < \lambda_{\alpha+1} < 1 \) and \( p > 1 \). Denote

\[
\theta_3 := \begin{cases} 
2^\frac{\nu - \lambda_{\alpha+1}}{\lambda_\alpha} - 1, & \text{if } \lambda_\alpha \geq \lambda_{\alpha+1}, \\
1, & \text{if } \lambda_\alpha \leq \lambda_{\alpha+1},
\end{cases} \tag{2.147}
\]

\[
L(x) := \left( 2 \int_{x_0}^x (q(w))^{\frac{1}{\nu - \lambda_{\alpha+1}}} \, dw \right)^{(1 - \lambda_{\alpha+1})} \left( \frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}} \right)^\lambda_{\alpha+1}, \tag{2.148}
\]

and

\[
P_1(x) := \int_{x_0}^x (x - t)^{\frac{\nu - \lambda_{\alpha+1}}{\nu - \gamma_1}} (p(t))^{-\frac{1}{\nu - \gamma_1}} \, dt, \tag{2.149}
\]

\[
T(x) := L(x) \cdot \left( \frac{P_1(x)^{\frac{\nu - 1}{\nu - \gamma_1}}}{\Gamma(\nu - \gamma_1)} \right)^{(\lambda_\alpha + \lambda_{\alpha+1})}, \tag{2.150}
\]

and

\[
\omega_1 := \begin{cases} 
2^\frac{\lambda_\alpha + \lambda_{\alpha+1}}{p}, & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \leq p, \\
1, & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \geq p,
\end{cases} \tag{2.151}
\]

\[
\Phi(x) := T(x) \omega_1.
\]

Also put

\[
\varepsilon_4 := \begin{cases} 
1, & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \geq p, \\
M^{1 - \frac{\lambda_\alpha + \lambda_{\alpha+1}}{p}}, & \text{if } \lambda_\alpha + \lambda_{\alpha+1} \leq p.
\end{cases} \tag{2.152}
\]

Then it holds

\[
\int_{x_0}^x q(w) \left\{ \sum_{j=1}^{M-1} \left[ \left| (D_{x_0}^{\gamma_1} f_j)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1+1} f_j+1)(w) \right|^{\lambda_{\alpha+1}} \\
+ \left| (D_{x_0}^{\gamma_1} f_{j+1})(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1+1} f_{j+1})(w) \right|^{\lambda_{\alpha+1}} \right] \right\} \\
+ \left[ \left| (D_{x_0}^{\gamma_1} f_1)(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1+1} f_{M})(w) \right|^{\lambda_{\alpha+1}} \right] \\
+ \left[ \left| (D_{x_0}^{\gamma_1} f_{M})(w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1+1} f_{1})(w) \right|^{\lambda_{\alpha+1}} \right] \right\} \, dw \\
\leq 2^\frac{\lambda_\alpha + \lambda_{\alpha+1}}{p} \varepsilon_4 \Phi(x) \left[ \int_{x_0}^x p(w) \left( \sum_{j=1}^{M} \left| (D_{x_0}^{\nu} f_j)(w) \right|^p \right) \, dw \right]^{\frac{\lambda_\alpha + \lambda_{\alpha+1}}{p}}, \tag{2.153}
\]

all \( x_0 \leq x \leq b \).

Similarly, by (2.153), we get
Theorem 2.38. Let all as in Assumptions 2.1. Here $\nu \geq 3$, $\gamma_1 \geq 1$ such that $\nu - \gamma_1 \geq 2$. Let $\lambda_\alpha > 0$, $0 < \lambda_{\alpha+1} < 1$, such that $p := \lambda_\alpha + \lambda_{\alpha+1} > 1$. Denote

$$\theta_3 := \begin{cases} 
2^{(\lambda_\alpha/\lambda_{\alpha+1})} - 1, & \text{if } \lambda_\alpha \geq \lambda_{\alpha+1}, \\
1, & \text{if } \lambda_\alpha \leq \lambda_{\alpha+1},
\end{cases} \quad (2.154)$$

$$L(R_2) := \left[ 2 \left( \frac{1 - \lambda_{\alpha+1}}{N - \lambda_{\alpha+1}} \right) \left( \frac{N - \lambda_{\alpha+1}}{N - \lambda_{\alpha+1}} \right) \right]^{1 - \lambda_{\alpha+1}} \left( \frac{\theta_3 \lambda_{\alpha+1}}{\lambda_\alpha + \lambda_{\alpha+1}} \right)^{\lambda_{\alpha+1}}, \quad (2.155)$$

and

$$P(R_2) := \int_{R_1}^{R_2} (R_2 - t)^{(\nu - \gamma_1 - 1)(\frac{p}{\nu})} t^{\frac{N - \nu}{\nu}} dt, \quad (2.156)$$

$$\Phi(R_2) := L(R_2) \left( \frac{P_1(R_2)(p-1)}{(\Gamma(\nu - \gamma_1))^p} \right). \quad (2.157)$$

Then

$$\int_A \left\{ \sum_{j=1}^{M-1} \left[ \frac{\partial^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}} \right]^{\lambda_\alpha} \left[ \frac{\partial^{\gamma_1+1} f_{j+1}(x)}{\partial r^{\gamma_1+1}} \right]^{\lambda_{\alpha+1}} + \left[ \frac{\partial^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}} \right]^{\lambda_\alpha} \left[ \frac{\partial^{\gamma_1+1} f_{j+1}(x)}{\partial r^{\gamma_1+1}} \right]^{\lambda_{\alpha+1}} \right\} + \left[ \frac{\partial^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right]^{\lambda_\alpha} \left[ \frac{\partial^{\gamma_1+1} f_1(x)}{\partial r^{\gamma_1+1}} \right]^{\lambda_{\alpha+1}}$$

$$\int_{x_0}^{x} q(w) \left\{ \sum_{j=1}^{M-1} \left[ (D_{x_0}^{\gamma_1} f_j)(w) \right]^{\lambda_\alpha} \left[ (D_{x_0}^{\gamma_1+1} f_{j+1})(w) \right]^{\lambda_{\alpha+1}} \left[ (D_{x_0}^{\nu} f_j)(w) \right]^{\lambda_{\nu}} \right\} \quad (2.158)$$

We need

Theorem 2.39. (see Anastassiou [3]) All here as in Theorem 2.31. Consider the special case $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. Denote

$$\tilde{T}(x) := A_0(x) \left( \frac{\lambda_\nu}{\lambda_\alpha + \lambda_\nu} \right)^{\lambda_\nu} 2^{(\nu + 2\lambda_{\alpha+1})}, \quad (2.159)$$

$$\tilde{\varepsilon}_5 := \begin{cases} 
1, & \text{if } 2(\lambda_\alpha + \lambda_\nu) \geq p, \\
M^{1 - \left( \frac{2(\lambda_\alpha + \lambda_\nu)}{p} \right)}, & \text{if } 2(\lambda_\alpha + \lambda_\nu) \leq p
\end{cases} \quad (2.160)$$

Then it holds

$$\int_{x_0}^{x} q(w) \left\{ \sum_{j=1}^{M-1} \left[ (D_{x_0}^{\gamma_1} f_j)(w) \right]^{\lambda_\alpha} \left[ (D_{x_0}^{\gamma_1+1} f_{j+1})(w) \right]^{\lambda_{\alpha+1}} \left[ (D_{x_0}^{\nu} f_j)(w) \right]^{\lambda_{\nu}} \right\} \quad (2.158)$$
\[ + \left\{ \left( D_{x_0}^{\gamma_2} f_j \right)(w) \mid \left( D_{x_0}^{\gamma_1} f_{j+1} \right)(w) \mid \left( D_{x_0}^{\nu} f_{j+1} \right)(w) \mid \left( D_{x_0}^{\nu} f_{M} \right)(w) \right\} \sum_{j=1}^{M} \left( \left( D_{x_0}^{\nu} f_j \right)(w) \right)^p ) \right] dw \]

\[ \leq 2^{\left( \frac{2\lambda_\alpha + \lambda_\nu}{p} \right)} \tilde{T}(x) \left[ \int_{x_0}^{x} p(w) \left( \sum_{j=1}^{M} \left( \left( D_{x_0}^{\nu} f_j \right)(w) \right)^p \right) \right], \quad (2.161) \]

all \( x_0 \leq x \leq b. \)

Similarly, by (2.161), we have

**Theorem 2.40.** Here all as in Theorem 2.32. Consider the case \( \lambda_\beta = \lambda_\alpha + \lambda_\nu; \)
\( \lambda_\alpha \geq 0, \lambda_\nu > 0, \lambda_\beta > 1/2, p := 2\lambda_\beta. \) Here \( P_k, k = 1, 2, \) as in (2.125) and \( A \) as in (2.126). Set

\[ A_0(R_2) := \left( \int_{R_1}^{R_2} \left( A(w) \right)^p \left( \lambda_\nu \right)^{(2\lambda_\alpha + \lambda_\nu)} \right). \quad (2.162) \]

Also put

\[ \tilde{T}(R_2) := A_0(R_2) \left( \frac{\lambda_\nu}{\lambda_\beta} \right)^{(2\lambda_\alpha + \lambda_\nu)} \quad (2.163) \]

Then

\[ \int_{x_0}^{x} \left\{ \left( \sum_{j=1}^{M} \left( \frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial \gamma_1} \right)^{\lambda_\alpha} \right) \left( \frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial \gamma_2} \right)^{\lambda_\nu} \right\} \left( \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial \nu} \right)^{\lambda_\nu} \]

\[ \leq 2 \tilde{T}(R_2) \left[ \sum_{j=1}^{M} \left( \int_{A} \left( \frac{\partial_{R_1}^{\nu} f_j(x)}{\partial \nu} \right)^p dx \right) \right], \quad (2.164) \]

We need

**Theorem 2.41.** (see Anastassiou [3]) Let \( \nu, \gamma_1, \gamma_2 \geq 1, \) such that \( \nu - \gamma_1 \geq 1, \nu - \gamma_2 \geq 1 \) and \( f_j \in C_{x_0}^{\nu}([a, b]) \) with \( f_j^{(i)}(x_0) = 0, i = 0, 1, \ldots, n-1, \quad n := \lceil \nu \rceil, \quad j = 1, \ldots, M \in \mathbb{N}. \) Here, \( x, x_0 \in [a, b]: x \geq x_0. \) Consider \( p(x) \geq 0 \) continuous functions on \([x_0, b]. \) Let \( \lambda_\alpha, \lambda_\beta, \lambda_\nu \geq 0. \) Set

\[ \rho(x) := \frac{(x - x_0)^{\left( \nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\beta + 1 \right)} \left\| p(x) \right\|_{\infty} \quad \left( \nu \lambda_\alpha - \gamma_1 \lambda_\alpha + \nu \lambda_\beta - \gamma_2 \lambda_\beta + 1 \right) \Gamma(\nu - \gamma_1 + 1) \Gamma(\nu - \gamma_2 + 1) \lambda_\beta. \quad (2.165) \]
Then it holds
\[
\int_{x_0}^{x} p(w) \left\{ \left\{ \sum_{j=1}^{M-1} \left[ \left| (D^{\gamma_1}_{x_0} f_j) (w) \right|^{\lambda_1} \left| (D^{\gamma_2}_{x_0} f_{j+1}) (w) \right|^{\lambda_2} \left| (D^\nu_{x_0} f_j) (w) \right|^{\lambda_3} \right] \right\} + \left[ \left| (D^{\gamma_1}_{x_0} f_j) (w) \right|^{\lambda_1} \left| (D^{\gamma_2}_{x_0} f_{j+1}) (w) \right|^{\lambda_2} \left| (D^\nu_{x_0} f_j) (w) \right|^{\lambda_3} \right] \right\} \right\}
\leq \rho(x) \left\{ \sum_{j=1}^{M} \left\{ \left\| (D^\nu_{x_0} f_j) \right\|_{\infty}^{2(\lambda_1+\lambda_2)} + \left\| (D^\nu_{x_0} f_j) \right\|_{\infty}^{2\lambda_3} \right\} \right\},
\tag{2.166}
\]
all \( x_0 \leq x \leq b \).

Similarly, by (2.166), we have

**Theorem 2.42.** All as in Assumption 2.1. Let \( \gamma_1, \gamma_2 \geq 1 \); such that \( \nu - \gamma_1 \geq 1 \), \( \nu - \gamma_2 \geq 1 \), \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \). Set
\[
\rho(R_2) = \frac{R^{N-1}_2(R_2 - R_1)^{\nu (\lambda_1 - \gamma_1 \lambda_2 + \lambda_2 \gamma_2 \lambda_3 + 1)} (\Gamma (\nu - \gamma_1 + 1))^{\lambda_0} (\Gamma (\nu - \gamma_2 + 1))^{\lambda_3}. \tag{2.167}
\]
Then
\[
\int_{A} \left\{ \left\{ \sum_{j=1}^{M-1} \left[ \left| \frac{\partial^{\gamma_1}_{R_1} f_j(x)}{\partial \tau^{\gamma_1}} \right|^{\lambda_1} \left| \frac{\partial^{\gamma_2}_{R_1} f_{j+1}(x)}{\partial \tau^{\gamma_2}} \right|^{\lambda_2} \left| \frac{\partial^\nu_{R_1} f_j(x)}{\partial \tau^\nu} \right|^{\lambda_3} \right] \right\} + \left[ \left| \frac{\partial^{\gamma_1}_{R_1} f_j(x)}{\partial \tau^{\gamma_1}} \right|^{\lambda_1} \left| \frac{\partial^{\gamma_2}_{R_1} f_{j+1}(x)}{\partial \tau^{\gamma_2}} \right|^{\lambda_2} \left| \frac{\partial^\nu_{R_1} f_j(x)}{\partial \tau^\nu} \right|^{\lambda_3} \right] \right\} \right\}
\leq \frac{2\pi^{N/2}}{\Gamma (N/2)} \rho(R_2) \left\{ \sum_{j=1}^{M} \left\{ \left\| \frac{\partial^\nu_{R_1} f_j}{\partial \tau^\nu} \right\|_{\infty}^{2(\lambda_1+\lambda_2)} + \left\| \frac{\partial^\nu_{R_1} f_j}{\partial \tau^\nu} \right\|_{\infty}^{2\lambda_3} \right\} \right\}. \tag{2.168}
\]

We need

**Theorem 2.43.** (see Anastassiou [3]) (As in Theorem 2.41, \( \lambda_3 = 0 \). It holds
\[
\int_{x_0}^{x} p(w) \left\{ \sum_{j=1}^{M} \left[ \left| (D^{\gamma_1}_{x_0} f_j) (w) \right|^{\lambda_1} \left| (D^\nu_{x_0} f_j) (w) \right|^{\lambda_3} \right] \right\} \right\}
\leq \left( \frac{(x - x_0)^{\nu \lambda_0 - \gamma_1 \lambda_1 + 1} \| p(x) \|_{\infty}}{(\nu \lambda_0 - \gamma_1 \lambda_1 + 1) (\Gamma (\nu - \gamma_1 + 1))^{\lambda_0}} \right) \cdot \left( \sum_{j=1}^{M} \left\| D^\nu_{x_0} f_j \right\|_{\infty}^{\lambda_0+\lambda_3} \right), \tag{2.169}
\]
all \( x_0 \leq x \leq b \).
Similarly, by (2.169), we obtain

**Theorem 2.44.** Here all as in Theorem 2.42. Case of $\lambda_\beta = 0$. Then

$$
\sum_{j=1}^{M} \left( \int_A \left| \frac{\partial_{\gamma_1} f_j(x)}{\partial r_{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{\gamma_2} f_j(x)}{\partial r_{\gamma_2}} \right|^{\lambda_\nu} dx \right)
\leq \left( \frac{2\pi^{N/2}}{\Gamma(N/2)} \right) \left( \frac{R_{2}^{N-1}(R_2 - R_1)^{(\nu\lambda_\alpha - \gamma_1\lambda_\alpha + 1)}(\Gamma(\nu - \gamma_1 + 1))}{\Gamma(\nu - \gamma_2 + 1))^{\lambda_\alpha}} \right) \left( \sum_{j=1}^{M} \left\| \frac{\partial_{\gamma_2} f_j}{\partial r_{\gamma_2}} \right\|_{\infty}^{\lambda_\alpha + \lambda_\nu} \right).
$$

(2.170)

We need

**Theorem 2.45.** (see Anastassiou [3]) (As in Theorem 2.41, $\lambda_\beta = \lambda_\alpha + \lambda_\nu$.) It holds

$$
\int_{x_0}^{x} p(w) \left\{ \sum_{j=1}^{M-1} \left[ \left| \frac{\partial_{\gamma_1} f_j}{\partial r_{\gamma_1}} (w) \right|^{\lambda_\alpha} \left| \frac{\partial_{\gamma_2} f_j}{\partial r_{\gamma_2}} (w) \right|^{\lambda_\nu} \left| \frac{\partial_{\gamma_1} f_{j+1}}{\partial r_{\gamma_1}} (w) \right|^{\lambda_\alpha} \left| \frac{\partial_{\gamma_2} f_{j+1}}{\partial r_{\gamma_2}} (w) \right|^{\lambda_\nu} \right] 
+ \left| \frac{\partial_{\gamma_2} f_j}{\partial r_{\gamma_2}} (w) \right|^{\lambda_\alpha + \lambda_\nu} \left| \frac{\partial_{\gamma_1} f_j}{\partial r_{\gamma_1}} (w) \right|^{\lambda_\alpha} \right\} dw
\leq \left( \frac{2(x - x_0)^{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)} \|p(x)\|_{\infty}}{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \right)
\cdot \left( \sum_{j=1}^{M} \left\| \frac{\partial_{\gamma_2} f_j}{\partial r_{\gamma_2}} \right\|_{\infty}^{(2\lambda_\alpha + \lambda_\nu)} \right).
$$

(2.171)

all $x_0 \leq x \leq b$.

Similarly, by (2.171), we derive

**Theorem 2.46.** Here all as in Theorem 2.42. Case of $\lambda_\beta = \lambda_\alpha + \lambda_\nu$. Then

$$
\int_A \left\{ \sum_{j=1}^{M-1} \left[ \left| \frac{\partial_{\gamma_1} f_j}{\partial r_{\gamma_1}} (x) \right|^{\lambda_\alpha} \left| \frac{\partial_{\gamma_2} f_{j+1}}{\partial r_{\gamma_2}} (x) \right|^{\lambda_\nu} \left| \frac{\partial_{\gamma_1} f_{j+1}}{\partial r_{\gamma_1}} (x) \right|^{\lambda_\alpha} \right] 
+ \left| \frac{\partial_{\gamma_2} f_j}{\partial r_{\gamma_2}} (x) \right|^{\lambda_\alpha + \lambda_\nu} \left| \frac{\partial_{\gamma_1} f_j}{\partial r_{\gamma_1}} (x) \right|^{\lambda_\alpha} \right\} dx
\leq \frac{4\pi^{N/2}}{\Gamma(N/2)} \cdot \left( \frac{R_{2}^{N-1}(R_2 - R_1)^{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)}(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}}{(2\nu\lambda_\alpha - \gamma_1\lambda_\alpha + \nu\lambda_\nu - \gamma_2\lambda_\alpha - \gamma_2\lambda_\nu + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha}} \right).
$$

(2.172)
\[ \frac{1}{(\Gamma(\nu - \gamma_2 + 1))(\lambda_\nu + \lambda_\omega)} \left( \sum_{j=1}^{M} \left\| \frac{\partial_{R_1}^\nu f_j}{\partial r^\nu} \right\|_\infty^{2(\lambda_\alpha + \lambda_\omega)} \right). \]  

(2.172)

We need

**Theorem 2.47.** (see Anastassiou [3]) (As in Theorem 2.41, \( \lambda_\nu = 0, \lambda_\alpha = \lambda_\beta \)). It holds

\[ \int_{x_0}^{x} p(w) \left\{ \sum_{j=1}^{M-1} \left[ \left| (D_{x_0}^{\gamma_1} f_j) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_2} f_{j+1}) (w) \right|^{\lambda_\alpha} \\
+ \left| (D_{x_0}^{\gamma_2} f_j) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1} f_{j+1}) (w) \right|^{\lambda_\alpha} \right] \right\} \\
+ \left\{ \left| (D_{x_0}^{\gamma_1} f_1) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_2} f_M) (w) \right|^{\lambda_\alpha} \right. \\
+ \left. \left| (D_{x_0}^{\gamma_2} f_1) (w) \right|^{\lambda_\alpha} \left| (D_{x_0}^{\gamma_1} f_M) (w) \right|^{\lambda_\alpha} \right\} \, dw \\
\leq 2 \rho^*(x) \left[ \sum_{j=1}^{M} \left\| D_{x_0}^{\nu} f_j \right\|_\infty^{2\lambda_\alpha} \right], \]  

(2.173)

all \( x_0 \leq x \leq b \). Here we have

\[ \rho^*(x) := \left( \frac{(x - x_0)^{(2\nu\lambda_\alpha - 1)(\lambda_\lambda - \gamma_1\lambda_\omega + 1)} \| p(x) \|_\infty}{(2\nu\lambda_\lambda - 1)(\lambda_\lambda - \gamma_1\lambda_\omega + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha} \Gamma(\nu - \gamma_2 + 1))^{\lambda_\omega}} \right). \]  

(2.174)

Similarly, by (2.174), we derive

**Theorem 2.48.** Here all as in Theorem 2.42. Case of \( \lambda_\nu = 0, \lambda_\alpha = \lambda_\beta \).

Then

\[ \int_{A} \left\{ \sum_{j=1}^{M-1} \left[ \left| \frac{\partial_{R_1}^{\gamma_1} f_j(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_{j+1}(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha} \\
+ \left| \frac{\partial_{R_1}^{\gamma_2} f_j(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1} f_{j+1}(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \right] \right\} \\
+ \left\{ \left| \frac{\partial_{R_1}^{\gamma_1} f_1(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_2} f_M(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha} \right. \\
+ \left. \left| \frac{\partial_{R_1}^{\gamma_2} f_1(x)}{\partial r^{\gamma_2}} \right|^{\lambda_\alpha} \left| \frac{\partial_{R_1}^{\gamma_1} f_M(x)}{\partial r^{\gamma_1}} \right|^{\lambda_\alpha} \right\} \, dx \\
\leq \left( \frac{4\pi^{N/2}}{\Gamma(N/2)} \right) \rho^*(R_2) \left[ \sum_{j=1}^{M} \left\| \frac{\partial_{R_1}^{\nu} f_j}{\partial r^{\nu}} \right\|_\infty^{2\lambda_\alpha} \right]. \]  

(2.175)

Here we have

\[ \rho^*(R_2) := \left( \frac{R_2^{N-1}(R_2 - R_1)^{(2\nu\lambda_\alpha - 1)(\lambda_\lambda - \gamma_1\lambda_\omega + 1)}}{(2\nu\lambda_\lambda - 1)(\lambda_\lambda - \gamma_1\lambda_\omega + 1)(\Gamma(\nu - \gamma_1 + 1))^{\lambda_\alpha} \Gamma(\nu - \gamma_2 + 1))^{\lambda_\omega}} \right). \]  

(2.176)

We need
Theorem 2.49. (see Anastassiou [3]) (As in Theorem 2.41, $\lambda_\alpha = 0$, $\lambda_\beta = \lambda_\nu$). It holds

$$\int_{x_0}^{x} p(w) \left\{ \sum_{j=1}^{M-1} \left[ \frac{1}{(D_{x_0}^\gamma f_j)(w)} \frac{1}{(D_{x_0}^\mu f_j)(w)} \right] \right. \right.$$

$$+ \left. \frac{1}{(D_{x_0}^\gamma f_j)(w)} \frac{1}{(D_{x_0}^\mu f_j)(w)} \right\} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \r
Hence, by ordinary integration by parts we have:
\[
\int_0^1 f(x) g^{(\nu)}(x) \, dx = \int_0^1 f(x) \, d \left( \mathcal{J}_{1-\alpha} \, g^{(n)} \right)(x)
\]
\[
= f(1) \left( \mathcal{J}_{1-\alpha} \, g^{(n)} \right)(1) - \int_0^1 \left( \mathcal{J}_{1-\alpha} \, g^{(n)} \right)(x) f'(x) \, dx,
\]
by \( \left( \mathcal{J}_{1-\alpha} \, g^{(n)} \right)(0) = 0 \).

Now we are ready to give

**Definition 2.3.** Let \( \nu > 0, \ n := [\nu], \ \alpha := \nu - n, \ g : [0, 1] \to \mathbb{R} \) such that there exists \( g^{(n)} \) which is measurable. Assume that \( \left( \mathcal{J}_{1-\alpha} \, g^{(n)} \right) \in L^1([0, 1]) \). We say that \( g^{(\nu)} \in L^1([0, 1]) \) is a **weak fractional derivative of order** \( \nu \) for \( g \), iff
\[
\int_0^1 u(x) \, g^{(\nu)}(x) \, dx = - \int_0^1 \left( \mathcal{J}_{1-\alpha} \, g^{(n)} \right)(x) u'(x) \, dx,
\]  
\[
\forall u \in C^\infty([0, 1]) : u(1) = 0.
\]  

Based on the above Definition 2.3, we can extend the concept of weak fractional differentiation to anchor points \( x_0 \neq 0 \), and to the multivariate case, especially to the radial case. Then try to generalize the results of this article.

**References**


