FIXED POINT RESULTS FOR MAPS WITH WEAKLY SEQUENTIALLY CLOSED GRAPHS

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ABSTRACT. In this paper we present an alternative of Leray-Schauder type and a fixed point result of Furi-Pera type. An application is given to illustrate our theory.

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1. INTRODUCTION

In this paper we first prove an alternative of Leray-Schauder type. This in particular improves a result in [5] where a condition was omitted. Then using this Leray-Schauder alternative we will obtain a new fixed point result of Furi-Pera type. This improves a result in [6] where one of the conditions was incorrectly stated and its proof needs to be adjusted slightly (see Theorem 2.4 below). Our results in particular extend those of [2, 3, 5, 12]. For the remainder of this section we gather some notations and preliminary facts. Let $X$ be a Banach space, let $B(X)$ denote the collection of all nonempty bounded subsets of $X$ and $W(X)$ the subset of $B(X)$ consisting of all weakly compact subsets of $X$. Also, let $B_r$ denote the closed ball centered at 0 with radius $r$.

Definition 1.1. A function $\psi : B(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

1. The family $\ker(\psi) = \{M \in B(X) : \psi(M) = 0\}$ is nonempty and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of $X$.
2. $M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.
3. $\psi(\overline{co}(M)) = \psi(M)$, where $\overline{co}(M)$ is the closed convex hull of $M$.
4. $\psi(\lambda M_1 + (1 - \lambda)M_2) \leq \lambda \psi(M_1) + (1 - \lambda)\psi(M_2)$ for $\lambda \in [0, 1]$.
5. If $(M_n)_{n \geq 1}$ is a sequence of nonempty weakly closed subsets of $X$ with $M_1$ bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n \to \infty} \psi(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.
The family ker $\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness $\psi$. Note that the intersection set $M_\infty$ from (5) belongs to ker $\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every $n$ and $\lim_{n \to \infty} \psi(M_n) = 0$. Also, it can be easily verified that the measure $\psi$ satisfies
\begin{equation}
\psi(\overline{M^w}) = \psi(M)
\end{equation}
where $\overline{M^w}$ is the weak closure of $M$.

A measure of weak noncompactness $\psi$ is said to be regular if
\begin{equation}
\psi(M) = 0 \text{ if and only if } M \text{ is relatively weakly compact,}
\end{equation}
subadditive if
\begin{equation}
\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2),
\end{equation}
homogeneous if
\begin{equation}
\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbb{R},
\end{equation}
set additive if
\begin{equation}
\psi(M_1 \cup M_2) = \max(\psi(M_1), \psi(M_2)).
\end{equation}

An important example of a measure of weak noncompactness has been defined by De Blasi [8] as follows:
\begin{equation}
w(M) = \inf\{r > 0 : \text{there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\},
\end{equation}
for each $M \in \mathcal{B}(X)$.

Notice that $w(\cdot)$ is regular, homogeneous, subadditive and set additive (see [8]).

Let $X$ and $Y$ be topological spaces. A multivalued map $F : X \to 2^Y$ is a point to set function if for each $x \in X$, $F(x)$ is a nonempty subset of $Y$. For a subset $M$ of $X$ we write $F(M) = \bigcup_{x \in M} F(x)$ and $F^{-1}(M) = \{x \in X : F(x) \cap M \neq \emptyset\}$. The graph of $F$ is the set $\text{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. We say that $F$ is upper semicontinuous (u.s.c. for short) at $x \in X$ if for every neighborhood $V$ of $F(x)$ there exists a neighborhood $U$ of $x$ with $F(U) \subseteq V$ (equivalently, $F : X \to 2^Y$ is u.s.c. if for any net $\{x_\alpha\}$ in $X$ and any closed set $B$ in $Y$ with $x_\alpha \to x_0 \in X$ and $F(x_\alpha) \cap B \neq \emptyset$ for all $\alpha$, we have $F(x_0) \cap B \neq \emptyset$). We say that $F : X \to 2^Y$ is upper semicontinuous if it is upper semicontinuous at every $x \in X$. The function $F$ is lower semicontinuous (l.s.c.) if the set $F^{-1}(B)$ is open for any open set $B$ in $Y$. If $F$ is l.s.c. and u.s.c., then $F$ is continuous.

If $Y$ is compact, and the images $F(x)$ are closed, then $F$ is upper semicontinuous if and only if $F$ has a closed graph. In this case, if $Y$ is compact, we have that $F$ is upper semicontinuous if $x_n \to x$, $y_n \to y$, and $y_n \in F(x_n)$, together imply that $y \in F(x)$. When $X$ is a Banach space we say that $F : X \to 2^X$ is weakly upper
semicontinuous if $F$ is upper semicontinuous in $X$ endowed with the weak topology. Also, $F : X \to 2^X$ is said to have weakly sequentially closed graph if the graph of $F$ is sequentially closed w.r.t. the weak topology of $X$.

**Definition 1.2.** Let $X$ be a Banach space and let $\psi$ be a measure of weak noncompactness on $X$. A multivalued mapping $B : D(B) \subseteq X \to 2^X$ is said to be $\psi$-condensing if it maps bounded sets into bounded sets and $\psi(B(S)) < \psi(S)$ whenever $S$ is a bounded subset of $D(B)$ such that $\psi(S) > 0$.

The following Sadovskii type fixed point theorem (see [5]) for multivalued mappings with weakly sequentially closed graph will be used in Section 2.

**Theorem 1.3.** Let $X$ be a Banach space, $\psi$ a regular set additive measure of weak noncompactness on $X$ and $C$ a nonempty closed convex subset of $X$. Suppose $F : C \to C(C)$ is $\psi$-condensing, $F(C)$ is bounded and $F$ has weakly sequentially closed graph; here $C(C)$ denotes the family of nonempty, closed, convex subsets of $C$. Then $F$ has a fixed point.

2. FIXED POINT THEOREMS

Our first result is a Leray-Schauder alternative principle.

**Theorem 2.1.** Let $X$ be a Banach space and $\psi$ a regular set additive measure of weak noncompactness on $X$. Let $Q$ and $C$ be closed, convex subsets of $X$ with $Q \subseteq C$. In addition, let $U$ be a weakly open subset of $Q$ with $0 \in U$. Suppose $F : U^w \to C(C)$ has weakly sequentially closed graph, $F(U^w)$ is bounded and $F$ is a $\psi$-condensing map. Also assume $U$ is weakly open in $C$ and $F$ transforms relatively weakly compact sets into relatively weakly compact sets. Then either

\[(2.1) \quad F \text{ has a fixed point,}\]

or

\[(2.2) \quad \text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda Fu;\]

where $\partial_Q U$ is the weak boundary of $U$ in $Q$.

**Proof.** Suppose (2.2) does not occur and $F$ does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.1) occurs). Let

$$M = \{x \in \overline{U^w} : x \in \lambda Fx \text{ for some } \lambda \in [0,1]\}.$$  

The set $M$ is nonempty since $0 \in U$. Also $M$ is weakly sequentially closed. Indeed let $(x_n)$ be sequence of $M$ which converges weakly to some $x \in \overline{U^w}$ and let $(\lambda_n)$ be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n Fx_n$. Then for each $n$ there is a $z_n \in Fx_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that $(\lambda_n)$
converges to some \( \lambda \in [0, 1] \) and without loss of generality assume \( \lambda_n \neq 0 \) for all \( n \).

This implies that the sequence \( (z_n) \) converges weakly to some \( z \in \overline{U^w} \) with \( x = \lambda z \).

Since \( F \) has weakly sequentially closed graph then \( z \in F(x) \). Hence \( x \in \lambda Fx \) and therefore \( x \in M \). Thus \( M \) is weakly sequentially closed. We now claim that \( M \) is relatively weakly compact. Suppose \( \psi(M) > 0 \). Since \( M \subseteq \text{co}(F(M) \cup \{0\}) \) then

\[
\psi(M) \leq \psi(\text{co}(F(M) \cup \{0\})) = \psi(F(M)) < \psi(M),
\]

which is a contradiction. Hence \( \psi(M) = 0 \) and therefore \( \overline{M^w} \) is weakly compact. This proves our claim. Now let \( x \in \overline{M^w} \). Since \( \overline{M^w} \) is weakly compact (Eberlein-Šmulian theorem [10 pg. 549]) then there is a sequence \( (x_n) \) in \( M \) which converges weakly to \( x \). Since \( M \) is weakly sequentially closed we have \( x \in M \). Thus \( \overline{M^w} = M \). Hence \( M \) is weakly closed and therefore weakly compact. From our assumptions we have \( M \cap \partial Q U = \emptyset \). Since \( X \) endowed with the weak topology is a locally convex space then there exists a continuous mapping \( \rho : \overline{U^w} \to [0, 1] \) with \( \rho(M) = 1 \) and \( \rho(\partial Q U) = 0 \).

Let

\[
J(x) = \begin{cases} 
\rho(x)F(x), & x \in \overline{U^w}, \\
0, & x \in C \setminus \overline{U^w}.
\end{cases}
\]

Clearly \( J : C \to C(C) \) has weakly sequentially closed graph since \( F \) has sequentially closed graph. Moreover, for any \( S \subseteq C \) we have

\[J(S) \subseteq \text{co}(J(S \cap U) \cup \{0\}).\]

If \( \psi(S \cap U) > 0 \) then

\[\psi(J(S)) \leq \psi(\text{co}(F(S \cap U) \cup \{0\})) = \psi(F(S \cap U)) < \psi(S \cap U) \leq \psi(S),\]

whereas if \( \psi(S \cap U) = 0 \) then

\[\psi(J(S)) \leq \psi(F(S \cap U)) = 0 < \psi(S),\]

if \( \psi(S) > 0 \). Thus \( J : C \to C(C) \) is \( \psi \)-condensing. From Theorem 1.3 there exists \( x \in C \) such that \( x \in J(x) \). Now \( x \in U \) since \( 0 \in U \). Consequently \( x \in \rho(x)F(x) \) and so \( x \in M \). This implies \( \rho(x) = 1 \) and so \( x \in F(x) \).

\[\square\]

**Remark 2.2.** In Theorem 2.1 above notice \( \partial Q U = \partial C U \). We note that the condition \( U \) is weakly open in \( C \) was omitted in Theorem 2.6 in [4] and in Theorem 2.1 (and the other results in Section 2) in [13] and the condition \( F \) transforms relatively weakly compact sets into relatively weakly compact sets was omitted in Theorem 2.2 in [5].

**Corollary 2.3.** Let \( X \) be a Banach space and \( \psi \) a regular set additive measure of weak noncompactness on \( X \). Let \( C \) be a closed, convex subsets of \( X \). In addition let \( U \) be a weakly open subset of \( C \) with \( 0 \in U \). Suppose \( F : \overline{U^w} \to C(C) \) has weakly sequentially closed graph, \( F(\overline{U^w}) \) is bounded and \( F \) is a \( \psi \)-condensing map. Also
assume $F$ transforms relatively weakly compact sets into relatively weakly compact sets. Then either

\begin{equation}
F \text{ has a fixed point,}
\end{equation}

or

\begin{equation}
\text{there is a point } u \in \partial_C U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda Fu;
\end{equation}

here $\partial_C U$ is the weak boundary of $U$ in $C$.

Our next result is a Furi-Pera type result.

**Theorem 2.4.** Let $X$ be a Banach space and $\psi$ a regular and set additive measure of weak noncompactness on $X$. Let $C$ be a closed convex subset of $X$ and $Q$ a closed convex subset of $C$ with $0 \in Q$. Assume the weak topology on $C$ is metrizable. Also, assume $F : Q \to C(C)$ has weakly sequentially closed graph, $F$ is $\psi$-condensing map, $F(Q)$ bounded and $F$ transforms relatively weakly compact sets into relatively weakly compact sets. In addition, assume that the following conditions are satisfied:

(i) there exists a weakly continuous retraction $r : X \to Q$, with $r(D) \subseteq \overline{co}(D \cup \{0\})$ for any bounded subset $D$ of $X$ and $r(x) = x$ for $x \in Q$.

(ii) if $\{(x_j, \lambda_j)\}_{j=1}^\infty$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial Q$, $\lambda_j \to \lambda$ and $x \in \lambda F(x)$, $0 \leq \lambda < 1$, then $\lambda_j F(x_j) \subseteq Q$ for $j$ sufficiently large; here $\partial Q$ is the weak boundary of $Q$ relative to $C$.

Then $F$ has a fixed point in $Q$.

**Proof.** Let $r$ be as described in (i) and let

\[ B = \{ x \in X : x \in Fr(x) \}. \]

We first show that $B \neq \emptyset$. To see this, consider $Fr : C \to C(C)$. Clearly $Fr$ has weakly sequentially closed graph, since $F$ has weakly sequentially closed graph and $r$ is weakly continuous. Now we show that $Fr$ is a $\psi$-condensing map. To see this, let $A$ be a bounded subset of $C$ and $\psi(A) > 0$. Now

\[ Fr(A) \subseteq Fr(\overline{co}(A \cup \{0\})). \]

Note $\psi(\overline{co}(A \cup \{0\})) = \psi(A) > 0$ so

\[ \psi(Fr(A)) < \psi(\overline{co}(A \cup \{0\})) = \psi(A). \]

Thus $Fr$ is a $\psi$-condensing map. Now Theorem 1.3 guarantees that there exists $y \in C$ with $y \in Fr(y)$. Thus $y \in B$ and $B \neq \emptyset$. Note $B$ is weakly sequentially closed, since $Fr$ has weakly sequentially closed graph. Moreover, we claim that $B$ is weakly compact. To see this, first note

\[ B \subseteq Fr(B) \subseteq Fr(\overline{co}(B \cup \{0\})). \]
If \( \psi(B) > 0 \) then since \( \psi(\overline{\psi}(B \cup \{0\})) = \psi(B) > 0 \) we have

\[
\psi(B) \leq \psi(F \overline{\psi}(B \cup \{0\})) < \psi(\overline{\psi}(B \cup \{0\})) = \psi(B),
\]

a contradiction. Thus, \( \psi(B) = 0 \) and so \( B \) is relatively weakly compact. Now let \( x \in \overline{B^w} \). Since \( \overline{B^w} \) is weakly compact then there is a sequence \( (x_n) \) of elements of \( B \) which converges weakly to some \( x \). Since \( B \) is weakly sequentially closed then \( x \in B \). Thus, \( \overline{B^w} = B \). This implies that \( B \) is weakly compact.

We now show that \( B \cap Q \neq \emptyset \). Suppose \( B \cap Q = \emptyset \). From our assumption the weak topology on \( C \) is metrizable, let \( d^* \) denote the metric. With respect to \( (C, d^*) \) note \( Q \) is closed, \( B \) is compact, \( B \cap Q = \emptyset \) so there exists \( \epsilon > 0 \) with

\[
d^*(B, Q) = \inf\{d^*(x, y) : x \in B, y \in Q\} > \epsilon.
\]

For \( i \in \{1, 2 \ldots \} \), let

\[
U_i = \left\{ x \in C : d^*(x, Q) < \frac{\epsilon}{i} \right\}.
\]

For each \( i \in \{1, 2 \ldots \} \) fixed, \( U_i \) is open with respect to \( d^* \) and so \( U_i \) is weakly open in \( C \). Also

\[
\overline{U_i^w} = \overline{U_i^{d^*}} = \left\{ x \in C : d^*(x, Q) \leq \frac{\epsilon}{i} \right\} \text{ and } \partial U_i = \left\{ x \in C : d^*(x, Q) = \frac{\epsilon}{i} \right\}.
\]

Note \( \overline{U_i^w} \cap B = \emptyset \), so Corollary 2.3 (with \( F = Fr, U = U_i \)) guarantees that there exists \( y_i \in \partial U_i \) and \( \lambda_i \in (0, 1) \) with \( y_i \in \lambda_i Fr(y_i) \); note \( Fr \) transforms relatively weakly compact sets into relatively weakly compact sets since \( r \) is weakly continuous and \( F \) transforms relatively weakly compact sets into relatively weakly compact sets and note also (see above) that \( Fr \) is a \( \psi \)-condensing map. Note since \( y_i \in \partial U_i \) that \( \lambda_i Fr(y_i) \not\subseteq Q \). We now consider

\[
D = \{ x \in X : x \in \lambda Fr(x), \text{ for some } \lambda \in [0, 1] \}.
\]

Note

\[
D \subseteq \overline{\psi}(Fr D \cup \{0\}) \subseteq \overline{\psi}(F \overline{\psi}(D \cup \{0\})) \cup \{0\}
\]

so if \( \psi(D) > 0 \) then since \( \psi(\overline{\psi}(D \cup \{0\})) = \psi(D) \) we have

\[
\psi(D) \leq \psi(\overline{\psi}(F \overline{\psi}(D \cup \{0\})) \cup \{0\})) = \psi(F(\overline{\psi}(D \cup \{0\}))) < \psi(\overline{\psi}(D \cup \{0\})) = \psi(D),
\]

a contradiction. Thus \( \psi(D) = 0 \) so \( D \) is relatively weakly compact. The reasoning above implies that \( D \) is weakly compact. Then, up to a subsequence, we may assume that \( \lambda_i \to \lambda^* \in [0, 1] \) and \( y_i \to y^* \in Q \). Since \( F \) has weakly sequentially closed graph then \( y^* \in \lambda^* Fr(y^*) \). Note \( \lambda^* \neq 1 \) since \( B \cap Q = \emptyset \). From assumption (ii) it follows that \( \lambda_j Fr(y_j) \subseteq Q \) for \( j \) sufficiently large, which is a contradiction. Thus \( B \cap Q \neq \emptyset \), so there exists \( x \in Q \) with \( x \in Fr(x) \), i.e. \( x \in Fr(x) \). \( \square \)
**Remark 2.5.** One of the conditions in Theorem 2.3 in [6] was stated incorrectly and the proof has to be adjusted slightly (i.e. modify slightly the proof of Theorem 2.4 above).

Next we establish an existence principle for the operator equation

\[(2.5) \quad y(t) \in N y(t), \quad t \in [0, T) \quad (T > 0 \text{ fixed})\]

in \(C([0, T], \mathbb{R}^n)\). Our result extends a result in [12, Theorem 3.9] and in [3, Theorem 2.8] (we note that one of the assumption in [12] was stated incorrectly). Recall \(W^{k,p}([0, T], \mathbb{R}^n), \quad 1 \leq p < \infty\), denotes the space of functions \(u : [0, T] \to \mathbb{R}^n\) with \(u^{(k-1)} \in AC[0, T]\) and \(u^{(k)} \in L^p[0, T]\). Note \(W^{k,p}([0, T], \mathbb{R}^n)\) is reflexive if \(1 < p < \infty\). Also we let \(\| \cdot \|_\infty\) denote the usual supremum norm and \(\| \cdot \|_2\) the usual \(L^2\) norm.

**Theorem 2.6.** Suppose \(N : W^{1,2}([0, T], \mathbb{R}^n) \to K(W^{1,2}([0, T], \mathbb{R}^n))\) has weakly sequentially closed graph; here \(K(W^{1,2}([0, T], \mathbb{R}^n))\) denotes the family of nonempty, convex, weakly closed subsets of \(W^{1,2}([0, T], \mathbb{R}^n)\). In addition assume the following two conditions hold:

\[(2.6) \quad \begin{cases} \exists M_0 > 0 \text{ such that if } u \in W^{1,2}([0, T], \mathbb{R}^n) \text{ satisfies} \\ \quad u \in \lambda Nu \text{ for } 0 < \lambda < 1, \quad \text{then } \|u\|_\infty \neq M_0 \end{cases}\]

and

\[(2.7) \quad \begin{cases} \exists N_0 \geq M_0, \text{ and } \exists N_1 > 0 \text{ such that if } u \in W^{1,2}([0, T], \mathbb{R}^n) \\ \quad \text{with } \|u\|_\infty \leq M_0 \text{ and } \|u'\|_2 \leq N_1, \quad \text{then } \|Nu\|_\infty \leq N_0 \quad \text{and } \|Nu\|_2 \leq N_1. \end{cases}\]

Then (2.5) has a solution in \(W^{1,2}([0, T], \mathbb{R}^n)\).

**Proof.** Let \(E = W^{1,2}([0, T], \mathbb{R}^n)\),

\(C = \{ u \in W^{1,2}([0, T], \mathbb{R}^n) : \|u\|_\infty \leq N_0 \text{ and } \|u'\|_2 \leq N_1 \}\)

and

\(U = \{ u \in W^{1,2}([0, T], \mathbb{R}^n) : \|u\|_\infty < M_0 \text{ and } \|u'\|_2 \leq N_1 \}\).

Notice \(C\) is a convex, closed, bounded subset of \(E\). We first show \(U\) is weakly open in \(C\). To do this we will show that \(C \setminus U\) is weakly closed. Let \(x \in \overline{C \setminus U}^w\). Then there exists \(x_n \in C \setminus U\) (see [7 pp. 81, 9 pp. 93]) with \(x_n \rightharpoonup x\) (here \(W^{1,2}([0, T], \mathbb{R}^n)\) is endowed with the weak topology and \(\rightharpoonup\) denotes weak convergence). We must show \(x \in C \setminus U\). Now since the embedding \(j : W^{1,2}([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}^n)\) is completely continuous [1], there is a subsequence \(S\) of integers with

\(x_n \rightharpoonup x \text{ in } C([0, T], \mathbb{R}^n)\) and \(x'_n \rightharpoonup x'\) in \(L^2([0, T], \mathbb{R}^n)\)

as \(n \to \infty\) in \(S\). Also

\(\|x\|_\infty = \lim_{n \to \infty} \|x_n\|_\infty\) and \(\|x'\|_2 \leq \lim \inf \|x'_n\|_2 \leq N_1.\)
Note $M_0 \leq \|x\|_\infty \leq N_0$ since $M_0 \leq \|x_n\|_\infty \leq N_0$ for all $n$. As a result $x \in C \setminus U$, so $C \setminus U^w = C \setminus U$. Thus $U$ is weakly open in $C$. Also

$$\partial U = \{u \in C : \|u\|_\infty = M_0\} \quad \text{and} \quad \overline{U^w} = \{u \in C : \|u\|_\infty \leq M_0\}.$$ 

To see this let $x \in \overline{U^w}$. Then [7 pp. 81] guarantees that there exists $x_n \in U$ with $x_n \rightharpoonup x$. Essentially the same reasoning as above yields $\|x\|_\infty \leq M_0$ and $\|x_n\|_p \leq N_1$, so $\overline{U^w} \subseteq \{u \in C : \|u\|_\infty \leq M_0\}$. On the other hand if $x \in A = \{u \in C : \|u\|_\infty \leq M_0\}$ (note $A$ is closed), then there exists $x_n \in U$ with $x_n \rightharpoonup x$ in $W^{1,2}([0,T], \mathbb{R}^n)$, so in particular $x_n \rightarrow x$ in $W^{1,2}([0,T], \mathbb{R}^n)$. Thus $x \in \overline{U^w}$, so $\overline{U^w} = \{u \in C : \|u\|_\infty \leq M_0\}$.

Next note $C$ is weakly compact (note $W^{1,2}([0,T], \mathbb{R}^n)$ is reflexive), (2.7) guarantees that $N : \overline{U^w} \rightarrow C(C)$ and $N$ transforms relatively weakly sets into relatively weakly compact sets (note $N(\overline{U^w}) \subseteq C$ and $C$ is weakly compact). Also (2.6) guarantees that (2.4) is not true (note if there exists $x \in \partial U$ and $\lambda \in (0,1)$ with $x \in \lambda N x$ then $\|x\|_\infty = M_0$ since $x \in \partial U$ and $\|x\|_\infty \neq M_0$ from (2.6)). Corollary 2.3 guarantees that $N$ has a fixed point in $\overline{U^w}$. 

**Remark 2.7.** In Theorem 2.6 it is enough to assume $N : \overline{U^w} \rightarrow K(C)$ has weakly sequentially closed graph; here $U$ and $C$ are as described in the proof.

**Remark 2.8.** Indeed it is clear that there is an analogue of Theorem 2.6 where $W^{1,2}([0,T], \mathbb{R}^n)$ is replaced by $W^{k,p}([0,T], \mathbb{R}^n)$, here $1 < p < \infty$.

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