POSITIVE SOLUTIONS AND EIGENVALUE INTERVALS FOR DISCRETE BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we study the existence of positive solutions and characterize the eigenvalue intervals for discrete boundary value problems. Both second order problems and $p$-Laplacian problems are considered. The proof relies on a well-known fixed point theorem in cones.

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1. INTRODUCTION

The purpose of this paper is to obtain the existence of positive solutions to discrete boundary value problems. We consider the second order problem

\[
\begin{align*}
\Delta^2 y(k-1) + f(k, y(k)) &= 0, \quad k \in N, \\
y(0) &= 0, \quad y(T + 1) = 0,
\end{align*}
\]

and the one-dimensional $p$-Laplacian problem

\[
\begin{align*}
\Delta(\phi(\Delta y(k-1))) + f(k, y(k)) &= 0, \quad k \in N, \\
y(0) &= 0, \quad y(T + 1) = 0,
\end{align*}
\]

where $T$ is a positive integer, $N$ is the discrete interval $\{1, 2, \ldots, T\}$, $\phi(s) = |s|^{p-2}s$, $p > 1$ and $\Delta y(k) = y(k + 1) - y(k)$ is the forward difference operator. Also we consider the second order system

\[
\begin{align*}
\Delta^2 y_i(k-1) + f_i(k, y(k)) &= 0, \quad k \in N, \\
y(0) &= 0, \quad y(T + 1) = 0, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

and the $p$-Laplacian system

\[
\begin{align*}
\Delta(\phi(\Delta y_i(k-1))) + f_i(k, y(k)) &= 0, \quad k \in N, \\
y(0) &= 0, \quad y(T + 1) = 0, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

where $y = (y_1, y_2, \ldots, y_n)$.

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Recently, discrete boundary value problems have been studied by many authors [1, 4, 10, 15]. In [4], positive solutions for nonpositone second order discrete problems were considered using a conical shell fixed point theorem. Multiple positive solutions of singular discrete problems were also studied in [6, 7] using variational methods. Moreover, discrete $p$–Laplacian problems have been considered in [9, 10, 15]. We refer the reader to [2] for a broad introduction to difference equations.

The main results in this paper are proved by employing a fixed point theorem (see Theorem 2.6) for compact maps on conical shells. To do this, we extend the ideas introduced by Lan in [18] to the discrete case. In the continuous case, this approach was also used to prove the existence of positive periodic solutions for systems of second order differential equations in [12]. The same problems for scalar functional differential equation were studied in [13]. The results obtained in [13] improve some results in [21]. Here we mention that the systems of integral equations or differential equations have been studied in [8, 14, 19, 20] by employing another fixed point theorem in cones (see Theorem 1.1 in [10]).

As applications, we characterize the eigenvalue intervals for problems

\[
\begin{align*}
\Delta^2 y(k - 1) + \lambda h(k)g(y(k)) &= 0, \ k \in N, \\
y(0) &= 0, \ y(T + 1) = 0,
\end{align*}
\]

and the $n$–dimensional systems

\[
\begin{align*}
\Delta^2 y_i(k - 1) + \lambda h_i(k)g_i(y(k)) &= 0, \ k \in N, \\
y(0) &= 0, \ y(T + 1) = 0, \\
y(T + 1) &= 0, \ i = 1, 2, \ldots, n.
\end{align*}
\]

Here $\lambda > 0$ is a positive parameter. We prove that the above eigenvalue problems have at least one positive solution for each $\lambda$ in an explicit eigenvalue interval. Recently, several eigenvalue characterizations for different kinds of boundary value problems have appeared and we refer the reader to [3, 5, 10, 11, 20].

The notation used is as follows. $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$, $N^+ = \{0, 1, \ldots, T + 1\}$. Recall that a map $f : N^+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous if it is continuous as a map of the topological space $N^+ \times \mathbb{R}_+$ into the topological space $\mathbb{R}_+$. In this paper, the topology on $N^+$ will be the discrete topology. Let $C(N^+, \mathbb{R})$ denote the class of map $y$ continuous on $N^+$ (discrete topology), with norm $\|y\|_0 = \max_{k \in N^+} |y(k)|$, then $C(N^+, \mathbb{R})$ is a Banach space. For any $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n$, $\|y\| = \max_{i=1,2,\ldots,n} |y_i|$.

The remaining part of the paper is organized as follows. In Section 2 some preliminary results will be given. In Section 3, we study the existence of positive solutions
for problem (1.1) and characterize the eigenvalue intervals for (1.5). Positive solutions of the second order system (1.3) and eigenvalue intervals for (1.7) are considered in section 4. In section 5, we study the existence of positive solutions for problem (1.2) and characterize the eigenvalue intervals for (1.6). Finally, section 6 studies the $p$-Laplacian system (1.4) and (1.8). It is worth remarking here that, Lemma 3.2 and Lemma 4.2 play a fundamental role in the proof of our main results.

2. PRELIMINARIES

In this section, we present some results, which will be needed in the following four sections. First we state some well-known results.

**Lemma 2.1** ([4]). Let $u \in C(N^+, \mathbb{R})$ satisfy $u(k) \geq 0$ for $k \in N^+$. If $y \in C(N^+, \mathbb{R})$ satisfies
\[
\begin{cases}
\Delta^2 y(k-1) + u(k) = 0, & k \in N, \\
y(0) = 0, & y(T+1) = 0,
\end{cases}
\]
then
\[y(k) \geq q(k)\|y\| \text{ for } k \in N^+;\]
here
\[q(k) = \min\left\{\frac{T+1-k}{T+1}, \frac{k}{T}\right\}. \tag{2.1}\]

**Remark 2.2.** From the definition of $q(k)$, we have $q(k) \geq \frac{1}{T+1}$ for $k \in N$.

**Lemma 2.3** ([15]). If $y \in C(N^+, \mathbb{R})$ satisfies
\[
\begin{cases}
\Delta(\phi(\Delta y(k-1))) \leq 0, & k \in N, \\
y(0) = 0, & y(T+1) = 0,
\end{cases}
\]
then $y(k) \geq q(k)\|y\| \text{ for } k \in N^+$; here $q(k)$ is defined in Lemma 2.1.

**Lemma 2.4** ([15]). If $u, v \in C(N^+, \mathbb{R})$ satisfies
\[
\begin{cases}
\Delta(\phi(\Delta u(k-1))) \leq \Delta(\phi(\Delta v(k-1))), & k \in N, \\
u(0) \geq v(0), & u(T+1) \geq v(T+1),
\end{cases}
\]
then $u(k) \geq v(k)$ for $k \in N^+$.

As indicated in the introduction, the proof of our main results is based on a fixed point theorem for compact maps on conical shells. We recall the statement of this result below, after introducing the definition of a cone.

**Definition 2.5.** Let $X$ be a Banach space and $K$ be a closed, nonempty subset of $X$. $K$ is a cone if

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$.
(ii) $u, -u \in K$ implies $u = 0$. 
We also recall that a completely continuous operator means a continuous operator which transforms every bounded set into a relatively compact set. If $D$ is a subset $X$, we write $D_K = D \cap K$ and $\partial K D = (\partial D) \cap K$.

**Theorem 2.6** ([17]). Let $X$ be a Banach space and $K$ a cone in $X$. Assume $\Omega^1$, $\Omega^2$ are open bounded subsets of $X$ with $\Omega^1_K \neq \emptyset$, $\overline{\Omega^2} \subset \Omega^2_K$. Let

$$T : \overline{\Omega^2}_K \to K$$

be a continuous and compact operator such that

(i) $\|Tx\| \leq \|x\|$ for $x \in \partial K \Omega^1$, and

(ii) there exists $e \in K \setminus \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial K \Omega^2$ and all $\lambda > 0$.

Then $T$ has a fixed point in $\overline{\Omega^2}_K \setminus \Omega^1_K$. The same conclusion remains valid if (i) holds on $\partial K \Omega^2$ and (ii) holds on $\partial K \Omega^1$.

### 3. SECOND ORDER PROBLEMS

In this section we study the existence of positive solutions to second order discrete boundary value problem (1.1) and characterize the eigenvalue intervals for (1.5).

Define the operator

$$Ty(k) = \sum_{j=1}^{T} G(k,j) f(j,y(j));$$

here $G(k,j)$ is defined by

$$G(k,j) = \begin{cases} 
\frac{j(T+1-k)}{T+1}, & 0 \leq j \leq k-1, \\
\frac{k(T+1-j)}{T+1}, & k \leq j \leq T+1. 
\end{cases}$$

In order to apply Theorem 2.6, we take $X_1 = C(N^+,\mathbb{R})$ with norm $|y|_0 = \max_{k \in N^+} |y(k)|$. Define a cone $K_1$ in $X_1$ by

$$K_1 = \{y \in X_1 : y(k) \geq q(k)\|y\| \text{ for } k \in N^+\}.$$  

**Lemma 3.1** ([1]). Assume that $f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous. Then $T$ is well defined and maps $K_1$ into $K_1$. Moreover, $T$ is continuous and completely continuous.

Following the ideas in [13, 18], we define the open sets

$$\Omega^r = \{y \in X_1 : \min_{k \in N} y(k) < \frac{r}{T+1}\} \text{ and } B^r = \{y \in X_1 : |y|_0 < r\}.$$  

**Lemma 3.2.** $\Omega^r$, $B^r$ defined above have the following properties:

(a) $\Omega^r_{K_1}$ and $B^r_{K_1}$ are open relative to $K_1$.

(b) $B^r_{K_1} \subset \Omega^r_{K_1} \subset B^r_{K_1}$.  

(c) \( y \in \partial K_1, \Omega_r \) if and only if \( y \in K_1 \) and \( \min_{k \in N} y(k) = \frac{r}{T+1} \).

(d) If \( y \in \partial K_1, \Omega_r \), then \( \frac{r}{T+1} \leq y(k) \leq r \) for \( k \in N \) and \( |y|_0 \leq r \).

Proof. (a) holds since \( \min_{k \in N} y(k) \) is continuous (discrete topology). (c) is clear. Let \( y \in \partial K_1, \Omega_r \), so by (c), we have \( \min_{k \in N} y(k) = \frac{r}{T+1} \) and \( y(k) \geq q(k)|y|_0 \geq \frac{1}{T+1}|y|_0 \) for \( k \in N \). Thus \( \frac{1}{T+1}|y|_0 \leq \min_{k \in N} y(k) = \frac{r}{T+1} \), so \( |y|_0 \leq r \) and \( \frac{r}{T+1} \leq y(k) \leq r \) for \( k \in N \), i.e., (d) holds.

Finally we prove (b). Let \( y \in B^{r/T+1} \), then \( |y|_0 < \frac{r}{T+1} \), so \( \min_{k \in N} y(k) < \frac{r}{T+1} \) and \( y \in \Omega_{K_1}^r \). If \( y \in \Omega_{K_1}^r \), then \( \min_{k \in N} y(k) < \frac{r}{T+1} \) and \( y(k) \geq q(k)|y|_0 \geq \frac{1}{T+1}|y|_0 \) for \( k \in N \). This implies \( |y|_0 < r \), i.e., \( \Omega_{K_1}^r \subset B_{K_1}^r \). Hence (b) holds. \( \square \)

It is clear that the sets \( \Omega^r \) are unbounded sets for each \( r > 0 \), so we cannot use Theorem 2.6 with \( \Omega^r \). However we will be able to apply Theorem 2.6 taking into account that, for each \( \delta > r \), the following relations hold:

\[
\Omega_{K_1}^r = (\Omega^r \cap B_{K_1}^\delta) \quad \text{and} \quad \Omega_{K_1}^r = (\Omega^r \cap B_{K_1}^\delta). 
\]

The first equality follows immediately from Lemma 3.2 (b). For the second let \( y \in \Omega_{K_1}^r \), then from Lemma 3.2 (c), we have that

\[
\frac{1}{T+1}|y|_0 \leq \min_{k \in N} y(k) \leq \frac{r}{T+1} < \frac{\delta}{T+1},
\]

so \( y \in (\Omega^r \cap B_{K_1}^\delta) \). Now, since \( \Omega^r \) and \( B_{K_1}^\delta \) are open sets we have \( \Omega^r \cap B_{K_1}^\delta \subset \Omega^r \cap B_{K_1}^\delta \). Thus \( y \in (\Omega^r \cap B_{K_1}^\delta) \), and therefore \( \Omega_{K_1}^r \subset (\Omega^r \cap B_{K_1}^\delta) \). The reverse inclusion is trivial.

**Theorem 3.3.** Assume that \( f : N^+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous. Furthermore, it is assumed that the following two hypotheses hold:

\( (D_1) \) there exist a constant \( \alpha > 0 \) and a continuous function \( \psi : N \to (0, \infty) \) such that

\[ f(j, y) \geq \frac{1}{T+1}\alpha \psi(j), \quad \text{for all } j \in N, \quad \frac{\alpha}{T+1} \leq y \leq \alpha \]

and

\[ \min_{k \in N} \sum_{j=1}^{T} G(k, j) \psi(j) \geq 1; \]

\( (D_2) \) there exist a constant \( \beta > 0 \) and a continuous function \( \chi : N \to (0, \infty) \) such that

\[ f(j, y) \leq \beta \chi(j), \quad \text{for all } j \in N, \quad 0 < y \leq \beta \]

and

\[ \max_{k \in N} \sum_{j=1}^{T} G(k, j) \chi(j) \leq 1. \]
Then, the following results hold:

(a) if $\beta < \frac{\alpha}{T+1}$, then problem (1.1) has at least one positive solution $y$ satisfying

$$\beta \leq |y|_0 \leq \alpha;$$

(b) if $\alpha < \beta$, then problem (1.1) has at least one positive solution $y$ satisfying

$$\frac{1}{T+1} \alpha \leq |y|_0 \leq \beta.$$

Proof. Since $y = y(k)$ is a solution of (1.1) whenever $y$ is a fixed point of $T$, we only need to prove that $T$ has at least one positive fixed point. To do this, we show that:

(i) $|Ty|_0 \leq |y|_0$ for $y \in \partial K_1 B^\beta$, and

(ii) there exists $e \in K_1 \setminus \{0\}$ such that $y \neq Ty + \lambda e$, for all $y \in \partial K_1 \Omega^\alpha$ and all $\lambda > 0$.

We start with (i). Now for any $y \in \partial K_1 B^\beta$, we have $|y|_0 = \beta$. Then from (D2) we obtain, for each $k \in N$,

$$Ty(k) = \sum_{j=1}^{T} G(k,j)f(j,y(j)) \leq \beta \sum_{j=1}^{T} G(k,j)\chi(j) \leq \beta \max_{k \in N} \sum_{j=1}^{T} G(k,j)\chi(j) \leq \beta.$$

Hence, $|Ty|_0 \leq |y|_0$ for each $y \in \partial K_1 B^\beta$ and so (i) holds.

Next we consider part (ii). Let $e(t) \equiv 1$, so $e \in K_1 \setminus \{0\}$. Next, suppose that there exists $y \in \partial K_1 \Omega^\alpha$ and $\lambda > 0$ such that $y = Ty + \lambda e$. Since $y \in \partial K_1 \Omega^\alpha$, then from Lemma 3.2 (d), we have $\frac{1}{T+1} \alpha \leq y(k) \leq \alpha$, $k \in N$.

From (D1) we have, for $k \in N$, that

$$y(k) = Ty(k) + \lambda = \sum_{j=1}^{T} G(k,j)f(j,y(j)) + \lambda \geq \frac{1}{T+1} \alpha \sum_{j=1}^{T} G(k,j)\psi(j) + \lambda \geq \frac{1}{T+1} \alpha \min_{k \in N} \sum_{j=1}^{T} G(k,j)\psi(j) + \lambda \geq \frac{1}{T+1} \alpha + \lambda.$$

Hence $\min_{k \in N} y(k) \geq \frac{1}{T+1} \alpha + \lambda > \frac{1}{T+1} \alpha$, contradicting the statement of Lemma 3.2 (c). This contradiction proves part (ii) above.

Now suppose that $\beta < \frac{1}{T+1} \alpha$. Then one has from Lemma 3.2 that $\overline{B^\beta K_1} \subset B^{\frac{\alpha}{T+1}} K_1 \subset \Omega^\alpha K_1$, and therefore it follows from Theorem 2.6 that $T$ has at least one fixed
point \( y \in \overline{\Omega^{\alpha K_1}} \setminus B^\beta_{K_1} \). Hence \(|y|_0 \geq \beta\) and \( \frac{1}{T+1} \beta \leq \min_{k \in N} y(k) \leq \frac{1}{T+1} \alpha \). On the other hand, \( \frac{1}{T+1} |y|_0 \leq \min_{k \in N} y(k) \leq \frac{1}{T+1} \alpha \). This implies that \(|y|_0 \leq \alpha\).

Finally, if \( \alpha < \beta \) one has \( \overline{\Omega^{\alpha K_1}} \subset B^\beta_{K_1} \), and then Theorem 2.6 guarantees the existence of at least one fixed point \( y \in B^\beta_{K_1} \setminus \Omega^{\alpha K_1}_1 \) of \( T \). Hence we obtain the inequality \( \frac{1}{T+1} \alpha \leq |y|_0 \leq \beta \). \( \Box \)

Next we employ Theorem 3.3 to establish the existence of positive solutions to the following discrete boundary value problem

\[
\begin{aligned}
\Delta^2 y(k-1) + h(k)g(y(k)) &= 0, \quad k \in N, \\
y(0) &= 0, \quad y(T+1) = 0.
\end{aligned}
\]  

(3.3)

We assume that

(H\(_1\)) \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is continuous with \( g(y) > 0 \) for \( y > 0 \).

(H\(_2\)) \( h(j) : N \rightarrow \mathbb{R}_+ \) is continuous and \( \sum_{j=1}^{T} G(k,j)h(j) > 0 \).

**Theorem 3.4.** Suppose that conditions (H\(_1\))-(H\(_2\)) hold. Then problem (3.3) has at least one positive solution \( y \) with \( y(k) \neq 0 \) for \( k \in N \) if one of the following conditions holds.

\( (h_1) \) \( 0 \leq g_0 < A^{-1} \) and \( B^{-1} < g_\infty \leq \infty \);

\( (h_2) \) \( 0 \leq g_\infty < A^{-1} \) and \( B^{-1} < g_0 \leq \infty \);

Here \( g_0 = \lim_{y \to 0^+} \frac{g(y)}{y} \), \( g_\infty = \lim_{y \to \infty} \frac{g(y)}{y} \), and

\[
A = \max_{k \in N} \sum_{j=1}^{T} G(k,j)h(j), \quad B = \min_{k \in N} \sum_{j=1}^{T} G(k,j)h(j).
\]

**Proof.** To see this, we will apply Theorem 3.3 with \( f(k,y) = h(k)g(y) \). We assume that \( (h_1) \) holds. The case when \( (h_2) \) holds is similar.

From the first part of \( (h_1) \), there exists \( \beta > 0 \) such that \( g(y) \leq A^{-1} \beta \) for \( 0 < y \leq \beta \). Choose \( \chi(k) = A^{-1}h(k) \). Then

\[
f(k,y) = h(k)g(y) \leq A^{-1} \beta h(k) = \beta \chi(k) \quad \text{if} \quad k \in N, \quad 0 < y \leq \beta,
\]

and

\[
\sum_{j=1}^{T} G(k,j)\chi(j) = A^{-1} \sum_{j=1}^{T} G(k,j)h(j) \leq A^{-1} \max_{k \in N} \sum_{j=1}^{T} G(k,j)h(j) = 1.
\]

Thus hypothesis \((D_2)\) holds.
From the second part of \((h_1)\), there exists \(\alpha > 0\) such that \(\frac{1}{T+1} \alpha > \beta\) and \(g(y) \geq B^{-1} \frac{1}{T+1} \alpha\) for \(y \geq \frac{1}{T+1} \alpha\). Choose \(\psi(k) = B^{-1} h(k)\), then

\[
 f(k, y) = h(k)g(y) \geq B^{-1} \frac{1}{T+1} \alpha h(k) = \frac{1}{T+1} \alpha \psi(k), \quad \text{if} \ k \in \mathbb{N}, \ y \geq \frac{1}{T+1} \alpha,
\]

(so in particular for \(\frac{1}{T+1} \alpha \leq y \leq \alpha\)) and

\[
 T \sum_{j=1}^{T} G(k, j) \psi(j) = B^{-1} \sum_{j=1}^{T} G(k, j) h(j) \geq B^{-1} \min_{k \in \mathbb{N}} \sum_{j=1}^{T} G(k, j) h(j) = 1.
\]

This implies that hypothesis \((D_1)\) holds. The result now follows from Theorem 3.3. □

By employing Theorem 3.4, we can easily characterize the eigenvalue intervals for problem \((1.5)\). Since the proof is easy, we only state the results here.

**Theorem 3.5.** Suppose that conditions \((H_1)-(H_2)\) hold. Then problem \((1.5)\) has at least one positive solution for each \(\lambda \in \left( \frac{1}{B g_\infty}, \frac{1}{A g_0} \right)\) (3.4) if \(\frac{1}{B g_\infty} < \frac{1}{A g_0}\). The same result remains valid for each \(\lambda \in \left( \frac{1}{B g_0}, \frac{1}{A g_\infty} \right)\) (3.5) if \(\frac{1}{B g_0} < \frac{1}{A g_\infty}\). Here we write \(1/g_\alpha = 0\) if \(g_\alpha = \infty\) and \(1/g_\alpha = \infty\) if \(g_\alpha = 0\), where \(\alpha = 0, \infty\).

### 4. SECOND ORDER SYSTEMS

In this section we establish the existence of positive solutions to the discrete system \((1.3)\) and characterize the eigenvalue intervals for \((1.7)\).

We take \(X_2 = C(N^+, \mathbb{R}) \times C(N^+, \mathbb{R}) \times \ldots \times C(N^+, \mathbb{R})\) \((n\) times\) with the norm \(\|y\| = \max_{i=1,2,\ldots,n} \|y_i\|\) for \(y = (y_1, y_2, \ldots, y_n) \in X_2\), here \(\|y\|_0 = \max_{i \in N^+} |y(i)|\). Then \(X_2\) is a Banach space. Define

\[
 K_2 = \{ y = (y_1, y_2, \ldots, y_n) \in X_2 : \min_{k \in N^+} y_i(k) \geq q(k) \|y_i\|_0 \text{ for all } i = 1, 2, \ldots, n \}. \quad (4.1)
\]

One can easily verify that \(K_2\) is a cone in \(X_2\). Moreover, let \(T : K_2 \to X_2\) be a map with components \((T^1, \ldots, T^n)\), where \(T^i, i = 1, 2, \ldots, n\), is defined by

\[
 (T_i y)(k) = \sum_{j=1}^{T} G(k, j) f^i(j, y(j)), \quad i = 1, 2, \ldots, n;
\]

here \(G(k, j)\) is given as \((3.1)\).
Lemma 4.1. Assume that \( f^i : N^+ \times \mathbb{R}^n_+ \to \mathbb{R}_+ \) is continuous, \( i = 1, 2, \ldots, n \). Then \( T \) is well defined and maps \( K_2 \) into \( K_2 \). Moreover, \( T \) is continuous and completely continuous.

Following the ideas in [12, 14], we define the open sets

\[
\Omega^r = \{ y \in X_2 : \min_{k \in N} y_i(k) < \frac{r}{T+1} \text{ for all } i = 1, 2, \ldots, n \}
\]

and

\[
B^r = \{ y \in X_2 : \|y\| < r \}.
\]

Lemma 4.2. \( \Omega^r \), \( B^r \) defined above have the following properties:

(a) \( \Omega_{K_2}^r \) and \( B_{K_2}^r \) are open relative to \( K_2 \).

(b) \( B_{K_2}^{r/T+1} \subset \Omega_{K_2}^r \subset B_{K_2}^r \).

(c) \( y \in \partial_{K_2} \Omega^r \) if and only if \( y \in K_2 \) and \( \min_{k \in N} y_j(k) = \frac{r}{T+1} \) for some \( j \in \{1, 2, \ldots, n\} \) and \( \min_{k \in N} y_i(k) \leq \frac{r}{T+1} \) for each \( i \in \{1, 2, \ldots, n\} \).

(d) If \( y \in \partial_{K_2} \Omega^r \), then \( \frac{r}{T+1} \leq y_j(k) \leq r \), \( k \in N \) for some \( j \in \{1, 2, \ldots, n\} \) and \( 0 \leq y_i(k) \leq r \), \( k \in N \) for each \( i \in \{1, 2, \ldots, n\} \). Moreover, \( |y_i|_0 \leq r \).

Proof. (a) is true since \( y_i(k) \) is continuous (discrete topology) for each \( i \in \{1, 2, \ldots, n\} \).

(c) is clear. Let \( y \in \partial_{K_2} \Omega^r \), so we have from (c) that there exists \( j \in \{1, 2, \ldots, n\} \) such that

\[
\frac{1}{T+1} \leq y_j(k) \leq r \quad \text{and} \quad \frac{r}{T+1} \leq y_j(k) \leq r, \quad k \in N.
\]

In addition notice for each \( i \in \{1, 2, \ldots, n\} \) that

\[
\frac{1}{T+1} \leq y_i(k) \leq \min_{k \in N} y_i(k) \leq \frac{r}{T+1}, \quad \text{so} \quad |y_i|_0 \leq r \quad \text{and} \quad 0 \leq y_i(k) \leq r \quad \text{for} \quad k \in N, \quad \text{i.e., (d) holds.}
\]

Finally we prove (b). Let \( y \in B_{K_2}^{r/T+1} \), then for each \( i \in \{1, 2, \ldots, n\} \), we have \( |y_i|_0 < \frac{r}{T+1} \), so \( \min_{k \in N} y_i(k) < \frac{r}{T+1} \) and \( y \in \Omega_{K_2}^r \). If \( y \in \Omega_{K_2}^r \), then for each \( i \in \{1, 2, \ldots, n\} \), we have \( \min_{k \in N} y_i(k) < \frac{r}{T+1} \) and \( y_i(k) \geq q(k) |y_i|_0 \geq \frac{1}{T+1} |y_i|_0 \) for \( k \in N \).

This implies \( |y_i|_0 < r \), i.e., \( \Omega_{K_2}^r \subset B_{K_2}^r \). Hence (b) holds. \( \square \)

Moreover, it is easy to verify that, for each \( \delta > r \), the following relations hold:

\[
\Omega_{K_2}^r = (\Omega^r \cap B^r)_{K_2} \quad \text{and} \quad \Omega_{K_2}^r = (\Omega^r \cap B^r)_{K_2}.
\]

Theorem 4.3. Assume that \( f^i : N^+ \times \mathbb{R}^n_+ \to \mathbb{R}_+ \) is continuous, \( i = 1, 2, \ldots, n \). Furthermore, it is assumed that the following two hypotheses hold:

(D3) For each \( i = 1, 2, \ldots, n \), there exist a constant \( \alpha > 0 \) and a continuous function \( \psi_i : N \to (0, \infty) \) such that

\[
f^i(j, y) \geq \frac{1}{T+1} \alpha \psi_i(j), \quad \text{for all} \quad j \in N, \quad 0 \leq y_i \leq \alpha (l \in \{1, 2, \ldots, n\} \setminus \{i\})
\]
and \( \frac{\alpha}{T+1} \leq y_i \leq \alpha \) and
\[
\min_{k \in \mathbb{N}} \sum_{j=1}^{T} G(k, j) \psi_i(j) \geq 1;
\]

**(D4)** For each \( i = 1, 2, \ldots, n \), there exist a constant \( \beta > 0 \) and a continuous function \( \chi_i : \mathbb{N} \to (0, \infty) \) such that
\[
f^i(j, y) \leq \beta \chi_i(j) \quad \text{for all } j \in \mathbb{N}, \ 0 < y_k \leq \beta, \ k \in \mathbb{N}
\]
and
\[
\max_{k \in \mathbb{N}} \sum_{j=1}^{T} G(k, j) \chi_i(j) \leq 1.
\]

Then, the following results hold:

(a) if \( \beta < \frac{\alpha}{T+1} \), then problem (1.3) has at least one positive solution \( y \) satisfying
\[
\beta \leq \|y\| = \max_{i \in \{1, \ldots, n\}} \max_{k \in \mathbb{N}} |y_i(k)| \leq \alpha;
\]

(b) if \( \alpha < \beta \), then problem (1.3) has at least one positive solution \( y \) satisfying
\[
\frac{1}{T+1} \alpha \leq \|y\| \leq \beta.
\]

**Proof.** We show that:

(i) \( \|Ty\| \leq \|y\| \) for \( y \in \partial K_2 B^\beta \), and

(ii) there exists \( e \in K_2 \setminus \{0\} \) such that \( y \neq Ty + \lambda e \), for all \( y \in \partial K_2 \Omega^\alpha \) and all \( \lambda > 0 \).

We start with (i). Now for any \( y \in \partial K_2 B^\beta \), we have \( |y_i|_0 \leq \beta \) for each \( i \in \{1, \ldots, n\} \). Fix \( i \in \{1, \ldots, n\} \). Then from (D4) we obtain, for each \( k \in \mathbb{N} \),
\[
(T_i y)(k) = \sum_{j=1}^{T} G(k, j) f^i(j, y(j)) \leq \beta \sum_{j=1}^{T} G(k, j) \chi_i(j)
\]
\[
\leq \beta \max_{k \in \mathbb{N}} \sum_{j=1}^{T} G(k, j) \chi_i(j) \leq \beta.
\]

Hence, \( |T_i y|_0 \leq \|y\| \) for each \( i \in \{1, \ldots, n\} \). This implies that (i) holds.

Next we consider part (ii). Let \( e(t) \equiv 1 \), so \( e \in K_2 \setminus \{0\} \). Next, suppose that there exists \( y \in \partial K_2 \Omega^\alpha \) and \( \lambda > 0 \) such that \( y = Ty + \lambda e \). Since \( y \in \partial K_2 \Omega^\alpha \), then from Lemma 4.2 (d) there exists \( i \in \{1, 2, \ldots, n\} \) with \( \frac{\alpha}{T+1} \leq y_i(k) \leq \alpha \), \( k \in \mathbb{N} \), and \( 0 \leq y_j(k) \leq \alpha \) for \( k \in \mathbb{N} \) and \( j \in \{1, 2, \ldots, n\} \setminus \{i\} \).
From (D₃) we have, for 𝑘 ∈ 𝑁, that

\[ y_𝑖(𝑘) = (𝑇_𝑖)𝑦(𝑘) + 𝜆 = \sum_{𝑗=1}^{𝑇} G(𝑘, 𝑗) f^i(𝑗, 𝑦(𝑗)) + 𝜆 \]

\[ \geq \frac{1}{𝑇 + 1} \alpha \sum_{𝑗=1}^{𝑇} G(𝑘, 𝑗)ψ_𝑗(𝑗) + 𝜆 \]

\[ \geq \frac{1}{𝑇 + 1} \alpha \min_{𝑘 ∈ 𝑁} \sum_{𝑗=1}^{𝑇} G(𝑘, 𝑗)ψ_𝑗(𝑗) + 𝜆 \geq \frac{1}{𝑇 + 1} \alpha + 𝜆. \]

Hence \( \min_{𝑘 ∈ 𝑁} y_𝑖(𝑘) \geq \frac{1}{𝑇 + 1} \alpha + 𝜆 > \frac{1}{𝑇 + 1} \alpha \), contradicting the statement of Lemma 4.2 (c). This contradiction proves part (ii) above.

Now suppose that \( β < \frac{α}{𝑇 + 1} \). Then one has from Lemma 4.2 that \( \overline{𝐵_β} 𝑂_{𝐾_2} \subset \overline{Ω_{𝐾_2}^α} \subset \Omega_{𝐾_2}^α \) and therefore it follows from Theorem 2.6 that \( 𝑇 \) has at least one fixed point \( y ∈ \overline{Ω_{𝐾_2}^α} \setminus \overline{𝐵_β} 𝑂_{𝐾_2} \). Hence \( ∥𝑦∥ ≥ β \) and \( \frac{1}{𝑇 + 1} β ≤ \min_{𝑘 ∈ 𝑁} y_𝑖(𝑘) ≤ \frac{α}{𝑇 + 1} \). On the other hand, \( \frac{1}{𝑇 + 1} |𝑦_0| ≥ \min_{𝑘 ∈ 𝑁} y_𝑖(𝑘) ≤ \frac{α}{𝑇 + 1} \) and therefore \( |𝑦_0| ≤ α \) for each \( 𝑖 ∈ \{1, 2, \ldots, 𝑛\} \). This implies that \( ∥𝑦∥ ≤ α \).

Finally, if \( α < β \) one has \( \overline{Ω_{𝐾_2}^α} \subset 𝐵_β 𝑂_{𝐾_2} \), and then Theorem 2.6 guarantees the existence of at least one fixed point \( y ∈ \overline{Ω_{𝐾_2}^α} \setminus 𝐵_β 𝑂_{𝐾_2} \) of \( 𝑇 \). Hence we obtain the inequality \( \frac{α}{𝑇 + 1} ≤ ∥y∥ ≤ β \). □

Next we employ Theorem 4.3 to establish the existence of positive solutions to the following second order discrete system

\[
\begin{align*}
\Delta^2 y_k(k-1) + h_k(k)y(k) &= 0, &k &\in \mathbb{N}, \\
y(0) &= 0, &y(T + 1) &= 0, &i &= 1, 2, \ldots, n.
\end{align*}
\]

(4.2)

We assume that

(H₃) \( g^i : \mathbb{R}^+_n \to \mathbb{R}^+_+ \) is continuous with \( g^i(y) > 0 \) for \( ∥y∥ > 0, \ i = 1, 2, \ldots, n \)

(H₄) \( h_k(i) : N \to \mathbb{R}^+_+ \) is continuous and \( \sum_{j=1}^{T} G(k, j)h_k(j) > 0, \ i = 1, 2, \ldots, n \)

Theorem 4.4. Suppose that conditions (H₃)–(H₄) hold. Then problem (4.2) has at least one positive solution \( y \) with \( y(k) \neq 0 \) for \( k ∈ 𝑁 \) if one of the following conditions holds.

(h₃) \( 0 ≤ g^i_0 < A_i^{-1} \) and \( B_i^{-1} < g^i_∞ ≤ \infty, \ i = 1, 2, \ldots, n; \)

(h₄) \( 0 ≤ g^i_∞ < A_i^{-1} \) and \( B_i^{-1} < g^i_0 ≤ \infty, \ i = 1, 2, \ldots, n; \)

Here \( g^i_0 = \lim_{y \to 0^+} \frac{g(y)}{∥y∥}, \ g^i_∞ = \lim_{y \to \infty} \frac{g(y)}{∥y∥}, \ i = 1, 2, \ldots, n, \) and

\[ A_i = \max_{k ∈ 𝑁} \sum_{j=1}^{T} G(k, j)h_k(j), \ B_i = \min_{k ∈ 𝑁} \sum_{j=1}^{T} G(k, j)h_k(j). \]
Proof. To see this, we will apply Theorem 4.3 with \( f^i(k, y) = h_i(k)g^i(y), i = 1, 2, \ldots, n \). We assume that \((h_3)\) holds. The case when \((h_4)\) holds is similar.

From the first part of \((h_3)\), there exists \( \beta > 0 \) such that \( g^i(y) \leq A_i^{-1}\beta \) for \( 0 < \|y\| \leq \beta \). Choose \( \chi_i(k) = A_i^{-1}h_i(k) \) for \( i = 1, 2, \ldots, n \). Fix \( i \in \{1, \ldots, n\} \). Then

\[
f^i(k, y) = h_i(k)g^i(y) \leq A_i^{-1}\beta h_i(k) = \beta \chi_i(k) \quad \text{if} \quad k \in N \text{ and } 0 < y_j \leq \beta,
\]

for \( j \in \{1, \ldots, n\} \) and

\[
\sum_{j=1}^{T} G(k, j)\chi_i(j) = A_i^{-1} \sum_{j=1}^{T} G(k, j)h_i(j) \leq A_i^{-1} \max_{k \in N} \sum_{j=1}^{T} G(k, j)h_i(j) = 1.
\]

Thus hypothesis \((D_4)\) holds.

From the second part of \((h_3)\), there exists \( \alpha > 0 \) such that \( \frac{1}{T+1} \alpha > \beta \) and \( g^i(y) \geq B_i^{-1}\frac{1}{T+1} \alpha \) for \( \|y\| \geq \frac{1}{T+1} \alpha \), \( i = 1, 2, \ldots, n \).

Thus \( g^i(y) \geq B_i^{-1}\frac{1}{T+1} \alpha \) for \( y_i \geq \frac{1}{T+1} \alpha \), \( i = 1, 2, \ldots, n \). Choose \( \psi_i(k) = B_i^{-1}h_i(k) \), then

\[
f^i(k, y) = h_i(k)g^i(y) \geq B_i^{-1}\frac{1}{T+1} \alpha h_i(k) = \frac{1}{T+1} \alpha \psi_i(k), \quad \text{if} \quad k \in N, \quad y_i \geq \frac{1}{T+1} \alpha,
\]

(so in particular for \( \frac{1}{T+1} \alpha \leq y_i \leq \alpha \)) and

\[
\sum_{j=1}^{T} G(k, j)\psi_i(j) = B_i^{-1} \sum_{j=1}^{T} G(k, j)h_i(j) \geq B_i^{-1} \min_{k \in N} \sum_{j=1}^{T} G(k, j)h_i(j) = 1.
\]

This implies that hypothesis \((D_3)\) holds. The result now follows from Theorem 4.3. \( \square \)

Now we characterize the eigenvalue intervals for the system \((1.7)\). Let

\[
A = \max\{A_i, i = 1, 2, \ldots, n\}, \quad B = \min\{B_i, i = 1, 2, \ldots, n\}.
\]

**Theorem 4.5.** Suppose that conditions \((H_3)-(H_4)\) hold. Then problem \((1.7)\) has at least one positive solution for each

\[
\lambda \in \left( \frac{1}{B \min_{i=1,2,\ldots,n} \{g^i_\infty\}}, \frac{1}{A \max_{i=1,2,\ldots,n} \{g^i_0\}} \right) \quad \text{(4.3)}
\]

if \( B \min_{i=1,2,\ldots,n} \{g^i_\infty\} < A \max_{i=1,2,\ldots,n} \{g^i_0\} \). The same result remains valid for each

\[
\lambda \in \left( \frac{1}{B \min_{i=1,2,\ldots,n} \{g^i_0\}}, \frac{1}{A \max_{i=1,2,\ldots,n} \{g^i_\infty\}} \right) \quad \text{(4.4)}
\]

if \( B \min_{i=1,2,\ldots,n} \{g^i_0\} < A \max_{i=1,2,\ldots,n} \{g^i_\infty\} \). Here we write \( 1/g^i_\alpha = 0 \) if \( g^i_\alpha = \infty \) and \( 1/g^i_\alpha = \infty \) if \( g^i_\alpha = 0 \), where \( \alpha = 0, \infty \).
Proof. We consider the case (4.3). The case (4.4) is similar. If $\lambda$ satisfies (4.3), then

$$\lambda g^i_0 \leq \lambda \max_{i=1,2,\ldots,n} \{g^i_0\} < \frac{1}{A_i}, \quad i = 1,2,\ldots,n,$$

and

$$\lambda g^i_\infty \geq \lambda \min_{i=1,2,\ldots,n} \{g^i_\infty\} > \frac{1}{B_i}, \quad i = 1,2,\ldots,n.$$

So Theorem 4.4 applies directly. □

5. ONE-DIMENSIONAL $p$-LAPLACIAN

In this section we establish the existence of positive solutions to the one-dimensional discrete $p$–Laplacian problem (1.2) and characterize the eigenvalue intervals for systems (1.6).

Define the operator

$$Ty(k) = \begin{cases} 0, & k = 0 \text{ or } T + 1, \\ T \sum_{s=k}^{T} \phi^{-1}(\tau + \sum_{r=1}^{s} f(r,u(r))), & k \in N, \end{cases}$$

where $\tau$ is a solution of the equation

$$\phi^{-1}(\tau) + T \sum_{k=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{k} f(r,u(r))) = 0.$$

Lemma 5.1 ([10]). Assume that $f : N^+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous. Then $T$ is well defined and maps $K_1$ into $K_1$. Moreover, $T$ is continuous and completely continuous; here $K_1$ is defined by (3.2).

Theorem 5.2. Assume that $f : N^+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous. Furthermore, it is assumed that the following two hypotheses hold:

(F1) there exist a constant $\alpha > 0$ and a continuous function $\psi : N \rightarrow (0,\infty)$ such that

$$f(j,y) \geq \phi(\frac{1}{T+1}\alpha)\psi(j), \quad \text{for all } j \in N, \quad \frac{\alpha}{T+1} \leq y \leq \alpha$$

and

$$\min_{k \in N} P(k) \geq 1;$$

here we assume $P(k)$ is the unique solution of

$$\begin{cases} \Delta(\phi(\Delta P(k-1))) + \psi(k) = 0, & k \in N, \\ P(0) = 0, \quad P(T + 1) = 0; \end{cases}$$
there exist a constant \( \beta > 0 \) and a continuous function \( \chi : \mathbb{N} \to (0, \infty) \) such that
\[
f(j, y) \leq \phi(\beta) \chi(j) \quad \text{for all } j \in \mathbb{N}, \ 0 < y \leq \beta
\]
and
\[
\max_{k \in \mathbb{N}} Q(k) \leq 1;
\]
here we assume \( Q(k) \) is the unique solution of
\[
\begin{cases}
\Delta(\phi(\Delta Q(k - 1))) + \chi(k) = 0, & k \in \mathbb{N}, \\
Q(0) = 0, & Q(T + 1) = 0.
\end{cases}
\]
Then, the following results hold:

(a) if \( \beta < \frac{\alpha}{T+1} \), then problem (1.2) has at least one positive solution \( y \) satisfying
\[
\beta \leq |y|_0 \leq \alpha;
\]

(b) if \( \alpha < \beta \), then problem (1.2) has at least one positive solution \( y \) satisfying
\[
\frac{1}{T+1} \alpha \leq |y|_0 \leq \beta.
\]

**Remark 5.3** ([7, 16]). For each continuous function \( \alpha : \mathbb{N} \to (0, \infty) \), the linear problem
\[
\begin{cases}
\Delta(\phi(\Delta y(k - 1))) + \alpha(k) = 0, & k \in \mathbb{N}, \\
y(0) = 0, & y(T + 1) = 0,
\end{cases}
\]
has exactly one positive solution. This guarantees the existence and uniqueness of \( P(k) \) and \( Q(k) \).

**Proof of Theorem 5.2.** We show that:

(i) \( |Ty|_0 \leq |y|_0 \) for \( y \in \partial K_1 B^\beta \), and
(ii) there exists \( e \in K_1 \setminus \{0\} \) such that \( y \neq Ty + \lambda e \), for all \( y \in \partial K_1 \Omega^\alpha \) and all \( \lambda > 0 \).

We start with (i). Now for any \( y \in \partial K_1 B^\beta \), we have \( |y|_0 = \beta \). Then from (F2), \( f(k, y) \leq \phi(\beta) \chi(k) \) for each \( k \in \mathbb{N} \). By Lemma 2.4, we obtain
\[
Ty(k) \leq \beta Q(k) \leq \beta \max_{k \in \mathbb{N}} Q(k) \leq \beta.
\]
Hence, \( |Ty|_0 \leq |y|_0 \) for each \( y \in \partial K_1 B^\beta \) and so (i) holds.

Next we consider part (ii). Let \( e(t) \equiv 1 \), so \( e \in K_1 \setminus \{0\} \). Next, suppose that there exists \( y \in \partial K_1 \Omega^\alpha \) and \( \lambda > 0 \) such that \( y = Ty + \lambda e \). Since \( y \in \partial K_1 \Omega^\alpha \), then from Lemma 3.2 (d), we have \( \frac{1}{T+1} \alpha \leq y(k) \leq \alpha \), \( k \in \mathbb{N} \).
From (F₁), \( f(k, y(k)) \geq \phi(\frac{1}{T+1}\alpha)\psi(k) \) for \( k \in N \). By Lemma 2.4, for each \( k \in N \), we have
\[
y(k) = Ty(k) + \lambda \geq \frac{1}{T+1}\alpha P(k) + \lambda \geq \frac{1}{T+1}\alpha \min_{k \in N} P(k) + \lambda \geq \frac{1}{T+1}\alpha + \lambda.
\]
Hence \( \min_{k \in N} y(k) \geq \frac{1}{T+1}\alpha + \lambda > \frac{1}{T+1}\alpha \), contradicting the statement of Lemma 3.2 (c). This contradiction proves part (ii) of our claim.

The rest of the proof is similar to that in the proof of Theorem 3.3, so we omit the details. □

Next we employ Theorem 5.2 to establish the existence of positive solutions to the following discrete problem
\[
\begin{cases}
\Delta(\phi(\Delta y(k - 1))) + h(k)g(y(k)) = 0, & k \in N, \\
y(0) = 0, & y(T + 1) = 0.
\end{cases}
\] (5.1)

We assume (H₁) holds. Moreover, it is assumed that
\( (H₃) \) \( H(k) > 0 \) for \( k \in N \); here we assume \( H(k) \) is the unique solution of
\[
\begin{cases}
\Delta(\phi(\Delta y(k - 1))) + h(k) = 0, & k \in N, \\
y(0) = 0, & y(T + 1) = 0.
\end{cases}
\]

**Theorem 5.4.** Suppose that conditions (H₁) and (H₃) hold. Then problem (5.1) has at least one positive solution \( y \) with \( y(k) \neq 0 \) for \( k \in N \) if one of the following conditions holds.

(i) \( 0 \leq g_0 < (\frac{1}{M})^{p-1} \) and \( (\frac{1}{m})^{p-1} < g_\infty \leq \infty \);

(ii) \( 0 \leq g_\infty < (\frac{1}{M})^{p-1} \) and \( (\frac{1}{m})^{p-1} < g_0 \leq \infty \);

here \( g_0 = \lim_{y \to 0^+} \frac{g(y)}{y^{p-1}} \), \( g_\infty = \lim_{y \to \infty} \frac{g(y)}{y^{p-1}} \), and
\[
M = \max_{k \in N} H(k), \quad m = \min_{k \in N} H(k).
\]

**Proof.** To see this, we will apply Theorem 5.2 with \( f(k, y) = h(k)g(y) \). We assume that (f₁) holds. The case when (f₂) holds is similar.

From the first part of (f₁), there exists \( \beta > 0 \) such that \( g(y) \leq (\frac{1}{M})^{p-1}\beta^{p-1} \) for \( 0 < y \leq \beta \). Choose \( \chi(k) = (\frac{1}{M})^{p-1}h(k) \). Then
\[
f(k, y) = h(k)g(y) \leq (\frac{1}{M})^{p-1}\beta^{p-1}h(k) = \beta^{p-1}\chi(k) \quad \text{if} \quad k \in N \quad \text{and} \quad 0 < y \leq \beta,
\]
and
\[
Q(k) = M^{-1}H(k) \leq M^{-1}\max_{k \in N} H(k) = 1.
\]
Thus hypothesis (F₂) holds.
From the second part of \((f_1)\), there exists \(\alpha > 0\) such that \(\frac{1}{T+1}\alpha > \beta\) and 
\[g(y) \geq \left(\frac{1}{m}\right)^{p-1} \left(\frac{1}{T+1}\right)^{p-1} h(k),\]
for \(y \geq \frac{1}{T+1}\alpha\). Choose \(\psi(k) = \left(\frac{1}{m}\right)^{p-1} h(k)\), then
\[f(k, y) = h(k) g(y) \geq \left(\frac{1}{m}\right)^{p-1} \left(\frac{1}{T+1}\right)^{p-1} h(k),\]
k \(\in\) \(\mathbb{N}\), \(y \geq \frac{1}{T+1}\alpha\), (so in particular for \(\frac{1}{T+1}\alpha \leq y \leq \alpha\)) and 
\[P(k) = m^{-1} H(k) \geq m^{-1} \min_{k \in \mathbb{N}} H(k) = 1.\]
This implies that hypothesis \((F_1)\) holds. The result now follows from Theorem 5.2. □

By employing Theorem 5.4, we can easily characterize the eigenvalue intervals
for \((1.6)\).

**Theorem 5.5.** Suppose that conditions \((H_1)\) and \((H_5)\) hold. Then problem \((1.6)\) has
at least one positive solution for each
\[\lambda \in \left(\frac{1}{m^{p-1}g_\infty}, \frac{1}{M^{p-1}g_0}\right)\]
if \(\frac{1}{m^{p-1}g_\infty} < \frac{1}{M^{p-1}g_0}\). The same result remains valid for each
\[\lambda \in \left(\frac{1}{m^{p-1}g_0}, \frac{1}{M^{p-1}g_\infty}\right)\]
if \(\frac{1}{m^{p-1}g_0} < \frac{1}{M^{p-1}g_\infty}\).

6. \(p\)-LAPLACIAN SYSTEMS

In this section we consider the discrete \(p\)-Laplacian system \((1.4)\) and \((1.8)\). Define
the operator \(T : X_2 \rightarrow X_2\) by
\[Ty = (T_1 y, T_2 y, \ldots, T_n y);\]
here
\[(T_i y)(k) = \begin{cases} 
0, & k = 0 \text{ or } T + 1, \\
\sum_{s=k}^{T} \phi^{-1}(\tau_i + \sum_{r=1}^{s} f_i(r, y(r))), & k \in \mathbb{N}, \quad i = 1, 2, \ldots, n,
\end{cases}\]
and \(\tau_i\) is a solution of the equation
\[\phi^{-1}(\tau) + \sum_{k=1}^{T} \phi^{-1}(\tau + \sum_{r=1}^{k} f_i(r, u(r))) = 0, \quad i = 1, 2, \ldots, n.\]

**Lemma 6.1.** Assume that \(f_i : N^+ \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+\) is continuous, \(i = 1, 2, \ldots, n\). Then
\(T\) is well defined and maps \(K_2\) into \(K_2\). Moreover, \(T\) is continuous and completely
continuous; here \(K_2\) is defined by \((4.1)\).

**Theorem 6.2.** Assume that \(f_i : N^+ \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+\) is continuous, \(i = 1, 2, \ldots, n\).
Furthermore, it is assumed that the following two hypotheses hold:
(F_3) For each \(i = 1, 2, \ldots, n\), there exist a constant \(\alpha > 0\) and a continuous function \(\psi_i : N \to (0, \infty)\) such that

\[
f^i(j, y) \geq \phi \left( \frac{1}{T + 1} \alpha \psi_i(j) \right), \quad \text{for all } j \in \mathbb{N}, \ 0 \leq y_i \leq \alpha \ (l \in \{1, 2, \ldots, n\} \setminus \{i\})
\]

and \(\frac{\alpha}{T+1} \leq y_i \leq \alpha\) and

\[
\min_{k \in \mathbb{N}} P_i(k) \geq 1;
\]

here we assume \(P_i(k)\) is the unique solution of

\[
\begin{cases}
\Delta (\phi (\Delta P(k - 1))) + \psi_i(k) = 0, \quad k \in \mathbb{N}, \\
P(0) = 0, \quad P(T + 1) = 0;
\end{cases}
\]

(F_4) For each \(i = 1, 2, \ldots, n\), there exist a constant \(\beta > 0\) and a continuous function \(\chi_i : N \to (0, \infty)\) such that

\[
f^i(j, y) \leq \phi(\beta) \chi_i(j) \quad \text{for all } j \in \mathbb{N}, \ 0 < y_k \leq \beta, \ k \in \mathbb{N}
\]

and

\[
\max_{k \in \mathbb{N}} Q_i(k) \leq 1;
\]

here we assume \(Q_i(k)\) is the unique solution of

\[
\begin{cases}
\Delta (\phi (\Delta Q(k - 1))) + \chi_i(k) = 0, \quad k \in \mathbb{N}, \\
Q(0) = 0, \quad Q(T + 1) = 0.
\end{cases}
\]

Then, the following results hold:

(a) if \(\beta < \frac{\alpha}{T+1}\), then problem (1.4) has at least one positive solution \(y\) satisfying

\[
\beta \leq \|y\| = \max_{i \in \{1, \ldots, n\}} \max_{k \in \mathbb{N}^+} |y_i(k)| \leq \alpha;
\]

(b) if \(\alpha < \beta\), then problem (1.4) has at least one positive solution \(y\) satisfying

\[
\frac{1}{T + 1} \alpha \leq \|y\| \leq \beta.
\]

Proof. We show that:

(i) \(\|Ty\| \leq \|y\|\) for \(y \in \partial K_2 B^\beta\), and

(ii) there exists \(e \in K_2 \setminus \{0\}\) such that \(y \neq Ty + \lambda e\), for all \(y \in \partial K_2 \Omega^\alpha\) and all \(\lambda > 0\).

We start with (i). Now for any \(y \in \partial K_2 B^\beta\), we have \(|y_i|_0 \leq \beta\) for each \(i \in \{1, 2, \ldots, n\}\). Fix \(i \in \{1, 2, \ldots, n\}\). Then from (F_4), \(f^i(k, y) \leq \phi(\beta) \chi_i(k)\) for each \(k \in \mathbb{N}\). By Lemma 2.4, we obtain

\[
(T_i y)(k) \leq \beta Q_i(k) \leq \beta \max_{k \in \mathbb{N}} Q_i(k) \leq \beta.
\]

Hence, \(|T_i y|_0 \leq \|y\|\) for each \(i \in \{1, 2, \ldots, n\}\). This implies that (i) holds.

Next we consider inequality (ii). Let \(e(t) \equiv 1\), so \(e \in K_2 \setminus \{0\}\). Next, suppose that there exists \(y \in \partial K_2 \Omega^\alpha\) and \(\lambda > 0\) such that \(y = Ty + \lambda e\). Since \(y \in \partial K_2 \Omega^\alpha\), then
from Lemma 4.2 (d), there exists \( i \in \{1, 2, \ldots, n\} \) with \( \frac{1}{T+1}\alpha \leq y_i(k) \leq \alpha, \ k \in N \) and \( 0 \leq y_j(k) \leq \alpha \) for \( j \in \{1, 2, \ldots, n\} \backslash \{i\} \).

From (F3), \( f^i(k, y(k)) \geq \phi \left( \frac{1}{T+1}\alpha \right) \psi_i(k) \) for \( k \in N \). By Lemma 2.4, for each \( k \in N \), we have

\[
y_i(k) = (T_iy)(k) + \lambda \geq \frac{1}{T+1}\alpha P_i(k) + \lambda \geq \frac{1}{T+1}\alpha \min_{k \in N} P_i(k) + \lambda \geq \frac{1}{T+1}\alpha + \lambda.
\]

Hence \( \min_{k \in N} y_i(k) \geq \frac{1}{T+1}\alpha + \lambda > \frac{1}{T+1}\alpha \), contradicting the statement of Lemma 4.2 (c). This contradiction proves part (ii) above.

The rest of the proof is similar to that in the proof of Theorem 4.3, so we omit the details. \( \square \)

Next we employ Theorem 5.2 to establish the existence of positive solutions to the following discrete system

\[
\begin{cases}
\Delta(\phi(\Delta y(k - 1))) + h_i(k)g^i(y(k)) = 0, & k \in N, \\
y(0) = 0, & y(T + 1) = 0.
\end{cases}
\]

We assume that (H3) holds. Moreover, it is assumed that

(\( H_6 \)) \( \widetilde{H}_i(k) > 0 \) for \( k \in N, \ i = 1, 2, \ldots, n; \) here we assume \( \widetilde{H}_i(k) \) is the unique solution of

\[
\begin{cases}
\Delta(\phi(\Delta y(k - 1))) + h_i(k) = 0, & k \in N, \\
y(0) = 0, & y(T + 1) = 0.
\end{cases}
\]

**Theorem 6.3.** Suppose that conditions (H3) and (H6) hold. Then problem (6.1) has at least one positive solution \( y \) with \( y(k) \neq 0 \) for \( k \in N \) if one of the following conditions holds.

(f3) \( 0 \leq g^i_0 < \left( \frac{1}{M_i}\right)^{p-1} \) and \( \left( \frac{1}{m_i}\right)^{p-1} < g^i_\infty \leq \infty, \ i = 1, 2, \ldots, n; \)

(f4) \( 0 \leq g^i_\infty < \left( \frac{1}{M_i}\right)^{p-1} \) and \( \left( \frac{1}{m_i}\right)^{p-1} < g^i_0 \leq \infty, \ i = 1, 2, \ldots, n; \)

here \( g^i_0 = \lim_{y \to 0^+} \frac{g^i(y)}{y^{p-1}}, \ g^i_\infty = \lim_{y \to \infty} \frac{g^i(y)}{y^{p-1}}, \) and

\[
M_i = \max_{k \in N} \widetilde{H}_i(k), \ m_i = \min_{k \in N} \widetilde{H}_i(k).
\]

**Proof.** To see this, we will apply Theorem 6.2 with \( f^i(k, y) = h_i(k)g^i(y), \ i = 1, 2, \ldots, n \). We assume that (f3) holds. The case when (f4) holds is similar.

From the first part of (f3), there exists \( \beta > 0 \) such that \( g^i(y) \leq \left( \frac{1}{M_i}\right)^{p-1}\beta^{p-1} \) for \( 0 < \|y\| \leq \beta \). Choose \( \chi_i(k) = \left( \frac{1}{M_i}\right)^{p-1}h_i(k) \) for \( i = 1, 2, \ldots, n \). Fix \( i \in \{1, \ldots, n\} \). Choose \( \chi_i(k) = \left( \frac{1}{M_i}\right)^{p-1} \). Then

\[
f^i(k, y) = h_i(k)g^i(y) \leq \left( \frac{1}{M_i}\right)^{p-1}\beta^{p-1}h_i(k) = \beta^{p-1}\chi_i(k) \quad \text{if} \ k \in N
\]
and $0 < y_j \leq \beta$, $j \in \{1, \ldots, n\}$ and
\[ Q_i(k) = M_i^{-1}\tilde{H}_i(k) \leq M_i^{-1}\max_{k \in N} \tilde{H}_i(k) = 1. \]

Thus hypothesis $(F_4)$ holds.

From the second part of $(f_3)$, there exists $\alpha > 0$ such that $\frac{1}{T+1}\alpha > \beta$ and $g^i(y) \geq (\frac{1}{m_i})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}$ for $\|y\| \geq \frac{1}{T+1}\alpha$, $i = 1, 2, \ldots, n$. Thus $g^i(y) \geq (\frac{1}{m_i})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}$ for $y_i \geq \frac{1}{T+1}\alpha$, $i = 1, 2, \ldots, n$. Fix $i \in \{1, \ldots, n\}$. Choose $\psi_i(k) = (\frac{1}{m_i})^{p-1}h_i(k)$, then
\[ f^i(k; y) = h_i(k)g^i(y) \geq (\frac{1}{m_i})^{p-1}(\frac{1}{T+1}\alpha)^{p-1}h_i(k) = (\frac{1}{T+1}\alpha)^{p-1}\psi_i(k), \quad k \in N, \]
$y \geq \frac{1}{T+1}\alpha$, (so in particular for $\frac{1}{T+1}\alpha \leq y \leq \alpha$) and
\[ P_i(k) = m_i^{-1}\tilde{H}_i(k) \geq m_i^{-1}\max_{k \in N} \tilde{H}_i(k) = 1. \]

This implies that hypothesis $(F_3)$ holds. The result now follows from Theorem 6.2. \(\Box\)

Finally we employ Theorem 6.3 we characterize the eigenvalue intervals for system (1.8). Here we only state the results and omit the proof since it is similar to that of Theorem 4.5.

Let
\[ M = \max\{M_i, i = 1, 2, \ldots, n\}, \quad m = \min\{m_i, i = 1, 2, \ldots, n\}. \]

**Theorem 6.4.** Suppose that conditions $(\text{H}_3)$ and $(\text{H}_6)$ hold. Then problem (1.8) has at least one positive solution for each
\[ \lambda \in \left(\frac{1}{M^{p-1}\max_{i = 1, 2, \ldots, n} \{g^i_0\}}, \frac{1}{M^{p-1}\min_{i = 1, 2, \ldots, n} \{g^i_\infty\}}\right) \]
if
\[ \frac{1}{M^{p-1}\min_{i = 1, 2, \ldots, n} \{g^i_\infty\}} < \frac{1}{M^{p-1}\max_{i = 1, 2, \ldots, n} \{g^i_0\}}. \]
The same result remains valid for each
\[ \lambda \in \left(\frac{1}{m^{p-1}\min_{i = 1, 2, \ldots, n} \{g^i_0\}}, \frac{1}{m^{p-1}\max_{i = 1, 2, \ldots, n} \{g^i_\infty\}}\right) \]
if
\[ \frac{1}{m^{p-1}\min_{i = 1, 2, \ldots, n} \{g^i_\infty\}} < \frac{1}{m^{p-1}\max_{i = 1, 2, \ldots, n} \{g^i_0\}}. \]

**REFERENCES**


