NONLINEAR INTEGRAL EQUATIONS IN BANACH SPACES AND HENSTOCK-KURZWEIL-PETTIS INTEGRALS

ANETA SIKORSKA-NOWAK

Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Umultowska 87, 61-614 Poznań, Poland (anetas@amu.edu.pl)

ABSTRACT. We prove an existence theorem for the nonlinear integral equation:

\[ x(t) = f(t) + \int_0^t k_1(t,s)x(s)ds + \int_0^t k_2(t,s)g(s,x(s))ds, \quad t \in I_\alpha = [0, \alpha], \quad \alpha \in \mathbb{R}_+, \]

with the Henstock-Kurzweil-Pettis integrals. This integral equation can be considered as a nonlinear Fredholm equation expressed as a perturbed linear equation. The assumptions about the function \( g \) are really weak: scalar measurability and weak sequential continuity with respect to the second variable. Moreover, we suppose that the function \( g \) satisfies some conditions expressed in terms of the measure of weak noncompactness.

Key words: existence of solution, Henstock-Kurzweil integral, Pettis integral, Henstock-Kurzweil-Pettis integral, nonlinear Fredholm integral equation, measures of weak noncompactness

AMS (MOS) Subject Classification: 34G20, 28B05, 45D05

1. INTRODUCTION

The Henstock-Kurzweil integral encompasses the Newton, Riemann and Lebesgue integrals [15, 19, 25]. A particular feature of this integral is that integrals of highly oscillating functions such as \( F'(t) \), where \( F(t) = t^2 \sin t^{-2} \) on \((0, 1]\) and \( F(0) = 0 \) can be defined. This integral was introduced by Henstock and Kurzweil independently in 1957-58 and has since proved useful in the study of ordinary differential equations [4, 8, 23, 24, 31]. In the paper [7] S. S. Cao defined the Henstock integral in a Banach space, which is a generalization of the Bochner integral. The Pettis integral is also a generalization of the Bochner integral [30]. This notion is strictly relative to weak topologies in Banach spaces.

In [10], we generalized both concepts of integral introducing the Henstock-Kurzweil-Pettis integral.

Let \((E, \|\cdot\|)\) be a Banach space, \(E^*\)- its dual space and \(I_\alpha = [0, \alpha], \alpha \in \mathbb{R}_+\)
Moreover, let \((C(I_\alpha, E), \omega)\) denote the space of all continuous functions from \(I_\alpha\) to \(E\) endowed with the topology \(\sigma(C(I_\alpha, E), C(I_\alpha, E)^*))\). In this paper we will prove an
existence theorem for the integral equation:

\[
(1) \quad x(t) = f(t) + \int_0^\alpha k_1(t, s)x(s)ds + \int_0^\alpha k_2(t, s)g(s, x(s))ds,
\]

where \( g : I_\alpha \times E \to E \), \( f : I_\alpha \to E \), \( x : I_\alpha \to E \) are functions with values in \( E \), \( k_1, k_2 : I_\alpha \times I_\alpha \to \mathbb{R}_+ \) and the integrals are taken in the sense of Henstock-Kurzweil-Pettis [11].

Note that the previous integral equation can be considered as a nonlinear Fredholm equation expressed as a perturbed linear equation.

We should mention that an extensive work has been done in the study of the solutions of particular cases of (1) (see, for example, [1, 2, 3, 20, 21, 26, 28, 29]).

The main result presented in this paper generalizes the previous ones.

A Kubiaczyk fixed point theorem [22] and the techniques of the theory of measure of weak noncompactness are used to prove the existence of solution of problem (1). The assumptions about the function \( g \) are really-weak: scalar measurability and weak sequential continuity with respect to the second variable. By using these conditions, we define a completely continuous operator \( F \) over the Banach space \( C([0, \alpha]) \), whose fixed points are solutions of (1). The fixed point theorem of Kubiaczyk [22] is used to prove the existence of a fixed point of the operator \( F \).

Let us recall, that a function \( f : I_\alpha \to E \) is said to be weakly continuous if it is continuous from \( I_\alpha \) to \( E \) endowed with its weak topology. A function \( g : E \to E_1 \), where \( E \) and \( E_1 \) are Banach spaces, is said to be weakly-weakly sequentially continuous if for each weakly convergent sequence \( (x_n) \) in \( E \), the sequence \( (g(x_n)) \) is weakly convergent in \( E_1 \). When the sequence \( x_n \) tends weakly to \( x_0 \) in \( E \), we will write \( x_n \overset{\omega}{\to} x_0 \).

Our fundamental tool is the measure of weak noncompactness developed by De-Blasi [6].

Let \( A \) be a bounded nonempty subset of \( E \). The measure of weak noncompactness \( \mu(A) \) is defined by

\[
\mu(A) = \inf\{t > 0 : \text{there exists } C \in K^\omega \text{ such that } A \subset C + tB_0\},
\]

where \( K^\omega \) is the set of weakly compact subsets of \( E \) and \( B_0 \) is the norm unit ball in \( E \).

We will use the following properties of the measure of weak noncompactness \( \mu \) (for bounded nonempty subsets \( A \) and \( B \) of \( E \)):

(i) if \( A \subset B \), then \( \mu(A) \leq \mu(B) \);
(ii) \( \mu(A) = \mu(\bar{A}) \), where \( \bar{A} \) denotes the closure of \( A \);
(iii) \( \mu(A) = 0 \) if and only if \( A \) is relatively weakly compact;
(iv) \( \mu(A \cup B) = \max \{\mu(A), \mu(B)\} \);
(v) \( \mu(\lambda A) = |\lambda|\mu(A) \), (\( \lambda \in \mathbb{R} \));
(vi) \( \mu(A + B) \leq \mu(A) + \mu(B) \);
(vii) \( \mu(\text{conv} A) = \mu(A) \).

It is necessary to remark that if \( \mu \) has these properties, then the following Lemma is true.

**Lemma 1.1** [27]. Let \( H \subset C(I_\alpha, E) \) be a family of strongly equicontinuous functions. Let, for \( t \in I_\alpha \), \( H(t) = \{h(t) \in E, \ h \in H\} \). Then \( \beta(H(I_\alpha)) = \sup_{t \in I_\alpha} \beta(H(t)) \) and the function \( t \mapsto \beta(H(t)) \) is continuous.

In the proof of the main result we will apply the following fixed point theorem.

**Theorem 1.2** [22]. Let \( X \) be a metrizable locally convex topological vector space. Let \( D \) be a closed convex subset of \( X \), and let \( F \) be a weakly sequentially continuous map from \( D \) into itself. If for some \( x \in D \) the implication

\[
\nabla = \overline{\text{conv}}(\{x\} \cup F(V)) \Rightarrow V \text{ is relatively weakly compact},
\]

holds for every subset \( V \) of \( D \), where \( \overline{\text{conv}}(\{x\} \cup F(V)) \) denotes the closure of the convex of \( \{x\} \cup F(V) \), then \( F \) has a fixed point.

Let us introduce the following definitions:

**Definition 1.3** [30]. Let \( G : [a, b] \to E \) and let \( A \subset [a, b] \). The function \( g : A \to E \) is a pseudoderivative of \( G \) on \( A \) if for each \( x^* \) in \( E^* \) the real-valued function \( x^*G \) is differentiable almost everywhere on \( A \) and \( (x^*G)' = x^*g \) almost everywhere on \( A \).

**Definition 1.4** [15, 25]. A family \( \mathcal{F} \) of functions \( F \) is said to be uniformly absolutely continuous in the restricted sense on \( X \) or, in short, uniformly AC\( s \) on \( X \) if for every \( \varepsilon > 0 \) there is \( \eta > 0 \) such that for every \( F \) in \( \mathcal{F} \) and for every finite or infinite sequence of non-overlapping intervals \( \{[a_i, b_i]\} \) with \( a_i, b_i \in X \) and satisfying \( \sum |b_i - a_i| < \eta \), we have \( \sum \omega(F, [a_i, b_i]) < \varepsilon \), where \( \omega(F, [a_i, b_i]) \) denotes the oscillation of \( F \) over \( [a_i, b_i] \) (i.e. \( \omega(F, [a_i, b_i]) = \sup \{|F(r) - F(s)| : r, s \in [a_i, b_i]\} \)).

A family \( F \) of functions \( F \) is said to be uniformly generalized absolutely continuous in the restricted sense on \( [a, b] \) or uniformly ACG\( s \) on \( [a, b] \) if \( [a, b] \) is the union of a sequence of closed sets \( A_i \) such that on each \( A_i \), the family \( F \) is uniformly AC\( s \) \((A_i)\).

**2. HENSTOCK-KURZWEIL-PETTIS INTEGRAL IN BANACH SPACES**

In this part we present the Henstock-Kurzweil-Pettis integral and we give properties of this integral.

**Definition 2.1** [15, 25]. Let \( \delta \) be a positive function defined on the interval \( [a, b] \).
A tagged interval \( (x, [c, d]) \) consists of an interval \( [c, d] \subseteq [a, b] \) and a point \( x \in [c, d] \).

The tagged interval \( (x, [c, d]) \) is subordinate to \( \delta \) if \( [c, d] \subseteq (x - \delta(x), x + \delta(x)) \).
Let \( P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n, \ n \in \mathbb{N}\} \) be such a collection in \([a, b]\). Then
(i) The points \( \{ s_i : 1 \leq i \leq n \} \) are called the tags of \( P \).
(ii) The intervals \( \{ [c_i, d_i] : 1 \leq i \leq n \} \) are called the intervals of \( P \).
(iii) If \( \{(s_i, [c_i, d_i]) : 1 \leq i \leq n \} \) is subordinate to \( \delta \) for each \( i \), then we write \( P \) is sub 
\( \delta \).
(iv) If \( [a, b] = \bigcup_{i=1}^{n} [c_i, d_i] \), then \( P \) is called a tagged partition of \( [a, b] \).
(v) If \( P \) is a tagged partition of \( [a, b] \) and if \( P \) is sub \( \delta \), then we write \( P \) is sub \( \delta \) on 
\( [a, b] \).
(vi) If \( f : [a, b] \to E \), then \( f(P) = \sum_{i=1}^{n} f(s_i)(d_i - c_i) \).
(vii) If \( F \) is defined on the subintervals of \( [a, b] \), then \( F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = 
\sum_{i=1}^{n} [F(d_i) - F(c_i)] \).

If \( F : [a, b] \to E \), then \( F \) can be treated as a function of intervals by defining 
\( F([c, d]) = F(d) - F(c) \). For such a function, \( F(P) = F(b) - F(a) \) if \( P \) is a tagged 
partition of \( [a, b] \).

**Definition 2.2** [15, 25]. A function \( f : [a, b] \to R \) is *Henstock-Kurzweil integrable on \( [a, b] \)* if there exists a real number \( L \) with the following property: for each \( \varepsilon > 0 \) there exists a positive function \( \delta \) on \( [a, b] \) such that \( |f(P) - L| < \varepsilon \) whenever \( P \) is a 
tagged partition of \( [a, b] \) that is subordinate to \( \delta \).

The function \( f \) is *Henstock-Kurzweil integrable on a measurable set \( A \subset [a, b] \)* if \( f_{\chi_A} \) is Henstock-Kurzweil integrable on \( [a, b] \). The number \( L \) in Definition 2.2 is called the *Henstock-Kurzweil integral of \( f \)* and we will denote it by \( \int_{a}^{b} f(t) \, dt \).

**Definition 2.3** [7]. A function \( f : [a, b] \to E \) is *Henstock-Kurzweil integrable on \( [a, b] \) \( (f \in HK([a, b], E)) \) if there exists a vector \( z \in E \) with the following property: for every \( \varepsilon > 0 \) there exists a positive function \( \delta \) on \( [a, b] \) such that \( \|f(P) - z\| < \varepsilon \) whenever \( P \) is a tagged partition of \( [a, b] \) sub \( \delta \). The function \( f \) is Henstock-Kurzweil integrable on a measurable set \( A \subset [a, b] \) if \( f_{\chi_A} \) is Henstock-Kurzweil integrable on 
\( [a, b] \). The vector \( z \) is the *Henstock-Kurzweil integral of \( f \).*

We remark that this definition includes the generalized Riemann integral defined by Gordon [16]. In a special case, when \( \delta \) is a constant function, we get the Riemann integral.

**Definition 2.4** [7]. A function \( f : [a, b] \to E \) is *HL integrable on \( [a, b] \) \( (f \in HL([a, b], E)) \) if there exists a function \( F : [a, b] \to E \), defined on the subintervals of 
\( [a, b] \), satisfying the following property: given \( \varepsilon > 0 \) there exists a positive function \( \delta \) on 
\( [a, b] \) such that if \( P = \{(s_i, [c_i, d_i]) : 1 \leq i \leq n \} \) is a tagged partition of \( [a, b] \) sub \( \delta \), then 
\[ \sum_{i=1}^{n} \|f(s_i)(d_i - c_i) - F([c_i, d_i])\| < \varepsilon. \]
Remark 1. We note that by triangle inequality:

\[ f \in HL([a, b], E) \quad \text{implies} \quad f \in HK([a, b], E). \]

In general, the converse is not true. For real-valued functions, the two integrals are equivalent.

**Definition 2.5** [30]. The function \( f : I_\alpha \to E \) is **Pettis integrable** (P integrable for short) if

(i) \( \forall x^* \in E^* \quad x^* f \) is Lebesgue integrable on \( I_\alpha \),
(ii) \( \forall A \subseteq I_\alpha, \text{A-measurable} \exists g \in E \ \forall x^* \in E^* \ x^* g = \left( L \right) \int_A x^* f(s)ds, \)

where \( (L) \int_A x^* f(s)ds \) denotes the Lebesgue integral over \( A \).

Now we present a definition of an integral which is a generalization for both: Pettis and Henstock-Kurzweil integrals.

**Definition 2.6** [11]. The function \( f : I_\alpha \to E \) is **Henstock-Kurzweil-Pettis integrable** (HKP integrable for short) if there exists a function \( g : I_\alpha \to E \) with the following properties:

(i) \( \forall x^* \in E^* \quad x^* f \) is Henstock-Kurzweil integrable on \( I_\alpha \) and
(ii) \( \forall t \in I_\alpha, \forall x^* \in E^* \ x^* g(t) = \left( HK \right) \int_0^t x^* f(s)ds. \)

This function \( g \) will be called a **primitive of** \( f \) and by \( g(\alpha) = \int_0^\alpha f(t)dt \) we will denote the **Henstock-Kurzweil-Pettis integral** of \( f \) on the interval \( I_\alpha \).

**Remark 2.** Each function which is HL integrable is integrable in the sense of Henstock-Kurzweil-Pettis. Our notion of integral is essentially more general than the previous ones (in Banach spaces):

(i) Pettis integral: by the definition of the Pettis integral and since each Lebesgue integrable function is HK integrable, a P integrable function is clearly HKP integrable.
(ii) Bochner, Riemann, and Riemann-Pettis integrals [16].
(iii) MsShane integral [14] or [17].
(iv) Henstock-Kurzweil (HL) integral ([7]).

We present below an example of a function which is HKP integrable but neither HL integrable nor P integrable.

**Example.** Let \( f : [0, 1] \to (L^\infty[0, 1], \| \cdot \|_\infty) \) be defined as \( f(t) = \chi_{[0,t]} + A(t) \cdot F'(t) \), where

\[
F(t) = t^2 \sin t^{-2}, \quad t \in (0, 1], \quad F(0) = 0, \quad \chi_{[0,t]}(\tau) = \begin{cases} 
1, & \tau \in [0, t], \\
0, & \tau \notin [0, t],
\end{cases} \quad t, \tau \in [0, 1],
\]

\( A(t)(\tau) = 1 \) for \( \tau, t \in [0, 1] \).
Put \( f_1(t) = \chi_{[0,t]} \), \( f_2(t) = A(t) \cdot F'(t) \).

We will show that the function \( f(t) = f_1(t) + f_2(t) \) is integrable in the sense of Henstock-Kurzweil-Pettis.

Observe that
\[
x^*(f(t)) = x^*(f_1(t) + f_2(t)) = x^*(f_1(t)) + x^*(f_2(t)).
\]
Moreover, the function \( x^*(f_1(t)) \) is Lebesgue integrable (in fact \( f_1 \) is Pettis integrable \([13]\)), so it is Henstock-Kurzweil integrable, and the function \( x^*(f_2(t)) \) is Henstock-Kurzweil integrable by Definition 2.2.

For each \( x^* \in E^* \) the function \( x^*f \) is not Lebesgue integrable because \( x^*f_2 \) is not Lebesgue integrable. So \( f \) is not Pettis integrable. Moreover, the function \( f_1 \) is not strongly measurable \([13]\) and the function \( f_2 \) is strongly measurable. So their sum \( f \) is not strongly measurable. Then by Theorem 9 from \([7]\) \( f \) is not HL integrable.

In this sequel we present some properties of the HKP integral which are important in the next part of our paper.

**Theorem 2.7** \([11]\). Let \( f : [a, b] \to E \) be HKP integrable on \([a, b]\) and let \( F(x) = \int_a^x f(s)ds, \ x \in [a, b]. \) Then

(i) for each \( x^* \in E^* \) the function \( x^*f \) is HK integrable on \([a, b]\) and
\[
(HK) \int_a^x x^*(f(s))ds = x^*(F(x))
\]
(ii) the function \( F \) is weakly continuous on \([a, b]\) and \( f \) is a pseudoderivative of \( F \) on \([a, b]\).

**Theorem 2.8** \([11]\). Let \( f : [a, b] \to E. \) If \( f = 0 \) almost everywhere on \([a, b]\), then \( f \) is HKP integrable on \([a, b]\) and \( \int_a^b f(t)dt = 0. \)

**Theorem 2.9** \([11]\) (Mean value theorem for the HKP integral). If the function \( f : I_a \to E \) is HKP integrable, then
\[
\int_I f(t)dt \in |I| \cdot \text{conv} f(I),
\]
where \( \text{conv} f(I) \) is the closure of the convex of \( f(I) \), \( I \) is an arbitrary subinterval of \( I_a \) and \( |I| \) is the length of \( I \).

**Theorem 2.10** \([9]\). Let \( f : I_a \to E \) and assume that \( f_n : I_a \to E, n \in N, \) are HKP integrable on \( I_a \). For each \( n \in N, \) let \( F_n \) be a primitive of \( f_n \). If we assume that:

(i) \( \forall x^* \in E^* \) \( x^*(f_n(t)) \to x^*(f(t)) \) a.e. on \( I_a, \)
(ii) for each \( x^* \in E^* \), the family \( G = \{ x^*F_n : n = 1, 2, \ldots \} \) is uniformly ACG on \( I_a \) (i.e. weakly uniformly ACG on \( I_a \)),
(iii) for each \( x^* \in E^* \), the set \( G \) is equicontinuous on \( I_a, \)
then $f$ is HKP integrable on $I_\alpha$ and $\int_0^t f(s)ds$ tends weakly in $E$ to $\int_0^t f(s)ds$ for each $t \in I_\alpha$.

3. EXISTENCE OF A SOLUTION

Now we prove the existence theorem for problem (1) under the weakest assumptions on $g$, as it is known.

For $x \in C(I_\alpha, E)$, we define the norm of $x$ by: $\|x\|_C = \sup \{ \|x(t)\|, \ t \in I_\alpha \}$.

Put $B = \{x \in C(I_\alpha, E) : x(0) = f(0), \ \|x\| \leq \|f(\cdot)\| + M, \ M > 0 \}$.

We define the operator $F : C(I_\alpha, E) \to C(I_\alpha, E)$ by

$$ F(x)(t) = f(t) + \int_0^\alpha k_1(t, s)x(s)ds + \int_0^\alpha k_2(t, s)g(s, x(s))ds, \ t \in I_\alpha, \ \alpha \in R_+, \ x \in B, $$

where integrals are taken in the sense of Henstock-Kurzweil-Pettis.

Moreover, let $\Gamma = \{F(x) \in C(I_\alpha, E) : x \in B\}$ and let $r(K)$ be the spectral radius of the integral operator $K$ defined by

$$ K(u)(t) = \int_0^\alpha [k_1(t, s) + k_2(t, s)]u(s)ds, \ t \in I_\alpha, \ u \in B. $$

Now we present the existence theorem for problem (1).

A continuous function $x : I_\alpha \to E$ is said to be a solution of problem (1) if it satisfies the equation (1) for every $t \in I_\alpha$.

**Theorem 3.1** Assume that for each continuous function $x : I_\alpha \to E$, $g(\cdot, x(\cdot))$ is HKP integrable, $g(s, \cdot)$ is weakly-weakly sequentially continuous and $k_1, k_2 : I_\alpha \times I_\alpha \to R_+$ are measurable functions such that $k_1(t, \cdot), k_2(t, \cdot)$ are continuous. Moreover, let $L > 0$ and

$$ (3) \quad \mu(g(I, X)) \leq L\mu(X) \quad \text{for each bounded subset} \ X \subset E, I \subset I_\alpha. $$

Suppose that $\Gamma$ is equicontinuous and uniformly $ACG_*$ on $I_\alpha$. Moreover, let $(1 + L)r(K) < 1$. Then there exists at least one solution of problem (1) on $I_\beta$, for some $0 < \beta \leq \alpha$, with continuous initial function $f$.

**Proof.** By equicontinuity of $\Gamma$ there exists some number $\beta$ ($0 < \beta \leq \alpha$) such that

$$ \| \int_0^\beta [k_1(t, s)x(s) + k_2(t, s)g(s, x(s))]ds \| \leq M \quad \text{for fixed} \ M > 0, \ t \in I_\beta \ \text{and} \ x \in B. $$

By our assumptions, the operator $F$ is well defined and maps $B$ into $B$. We will show that the operator $F$ is weakly sequentially continuous.

By Lemma 9 of [27], a sequence $x_n(\cdot)$ is weakly convergent in $C(I_\beta, E)$ to $x(\cdot)$ if and only if $x_n(t)$ tends weakly to $x(t)$ for each $t \in I_\beta$. Because $g(s, \cdot)$ is weakly-weakly
sequentially continuous, so if \( x_n \xrightarrow{\omega} x \) in \((C(I_\beta, E), \omega)\) then \( g(s, x_n(s)) \xrightarrow{\omega} g(s, x(s)) \) in \( E \) for \( t \in I_\beta \) and by Theorem 2.10 we have
\[
\lim_{n \to \infty} \int_{0}^{\beta} [k_1(t, s)x_n(s) + k_2(t, s)g(s, x_n(s))]ds = \int_{0}^{\beta} [k_1(t, s)x(s) + k_2(t, s)g(s, x(s))]ds
\]
weakly in \( E \), for each \( t \in I_\beta \). We see that \( F(x_n)(t) \to F(x)(t) \) weakly in \( E \) for each \( t \in I_\beta \) so \( F(x_n) \to F(x) \) in \((C(I_\beta, E), \omega)\).

Suppose that \( V \subset B \) satisfies the condition \( \bar{V} = \overline{\text{conv}(\{x\} \cup F(V))} \), for some \( x \in B \). We will prove that \( V \) is relatively weakly compact, thus (2) is satisfied.

Let, for \( t \in I_\beta \), \( V(t) = \{v(t) \in E : v \in V\} \).

From the definition of \( B \) and Lemma 1.1, it follows that the function \( v : t \mapsto \mu(V(t)) \) is continuous on \( I_\beta \).

We divide the interval \( I_\beta : 0 = t_0 < t_1 < \cdots < t_m = \beta \), where \( t_i = \frac{i\beta}{m} \), \( i = 0, 1, \ldots, m \). Let \( V([t_i, t_{i+1}]) = \{u(s) \in E : u \in V, t_i \leq s \leq t_{i+1}\} \), \( i = 0, 1, \ldots, m - 1 \). By Lemma 1.1 and the continuity of \( v \) there exists \( s_i \in T_i = [t_i, t_{i+1}] \) such that
\[
\mu(V([t_i, t_{i+1}])) = \sup\{\mu(V(s)) : t_i \leq s \leq t_{i+1}\} =: v(s_i).
\]

On the other hand, by the definition of the operator \( F \) and Theorem 2.11 we have
\[
F(u)(t) = f(t) + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} [k_1(t, s)u(s) + k_2(t, s)g(s, u(s))]ds
\]
\[
\in f(t) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{conv}}[k_1(t, T_i)V([t_i, t_{i+1}]) + k_2(t, T_i)g(T_i, V([t_i, t_{i+1}])])
\]
for each \( u \in V \).

Therefore
\[
F(V(t)) \subset f(t) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)\overline{\text{conv}}[k_1(t, T_i)V([t_i, t_{i+1}]) + k_2(t, T_i)g(T_i, V([t_i, t_{i+1}])])
\]

Using (3), (4) and the properties of the measure of weak noncompactness \( \mu \) we obtain
\[
\mu(F(V(t))) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i)k_1(t, T_i)\mu(V([t_i, t_{i+1}])
\]
\[
+ \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_2(t, T_i)\mu(g(T_i, V([t_i, t_{i+1}])))]
\]
\[
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_1(t, T_i)v(s_i) + k_2(t, T_i)Lv(s_i)]
\]
\[
= \sum_{i=0}^{m-1} (t_{i+1} - t_i)k_1(t, T_i)v(s_i) + L \sum_{i=0}^{m-1} (t_{i+1} - t_i)k_2(t, T_i)v(s_i)
\]
Therefore
\[
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \sup_{s \in T_i} k_1(t, s)v(s) + L \sum_{i=0}^{m-1} (t_{i+1} - t_i) \sup_{s \in T_i} k_2(t, s)v(s)
\]
\[
= \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_1(t, p_i)v(s_i) + L \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_2(t, q_i)v(s_i),
\]

where \(s_i, p_i, q_i \in T_i\), hence
\[
\mu(F(V(t))) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_1(t, p_i)v(p_i) + \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_1(t, p_i)(v(s) - v(p_i))]
\]
\[
+ L \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_2(t, q_i)v(q_i)
\]
\[
+ L \sum_{i=0}^{m-1} (t_{i+1} - t_i)[k_2(t, q_i)(v(s) - v(q_i))]
\]
\[
= \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_1(t, p_i)v(p_i) + \frac{\beta}{m} \sum_{i=0}^{m-1} [k_1(t, p_i)(v(s) - v(p_i))]
\]
\[
+ L \sum_{i=0}^{m-1} (t_{i+1} - t_i) k_2(t, q_i)v(q_i) + \frac{L \beta}{m} \sum_{i=0}^{m-1} [k_2(t, q_i)(v(s) - v(q_i))].
\]

By the continuity of \(v\) we have \(v(s_i) - v(p_i) < \varepsilon_1\) and \(\varepsilon_1 \to 0\) as \(m \to \infty\) and \(v(s_i) - v(q_i) < \varepsilon_2\) and \(\varepsilon_2 \to 0\) as \(m \to \infty\).

So
\[
\mu(F(V(t))) < \frac{\beta}{\int_0^\beta k_1(t, s)v(s)ds + \beta \sup_{p \in I_\beta} k_1(t, p) \varepsilon_1}
\]
\[
+ L \int_0^\beta k_2(t, s)v(s)ds + \frac{L \beta}{\int_0^\beta k_2(t, q)v(s)ds, \quad \text{for } \beta \in I_\beta}
\]

Therefore
\[
(5) \quad \mu(F(V(t))) \leq (1 + L) \int_0^\beta [k_1(t, s) + k_2(t, s)]v(s)ds, \quad \text{for } t \in I_\beta.
\]

Since \(V = \overline{\text{conv}}(\{u\} \cup F(V))\), by the property of the measure of weak noncompactness we have \(\mu(V(t)) \leq \mu(F(V(t)))\) and so in view of (5), it follows that \(\mu(t) \leq (1 + L) \int_0^\beta [k_1(t, s) + k_2(t, s)]v(s)ds, \quad \text{for } t \in I_\beta\). Because this inequality holds for every \(t \in I_\beta\) and \((1 + L)r(K) < 1\), so by applying Gronwall’s inequality [18], we conclude that \(\mu(V(t)) = 0\), for \(t \in I_\beta\). Hence Arzela-Ascoli’s theorem implies that the set \(V\) is relatively compact. Consequently, by Theorem 1.2, \(F\) has a fixed point which is a solution of the problem (1).
Remark 3. The condition (3) in our Theorem 3.1 can be also generalized to the Sadovskii conditions: $\mu(F(I \times X)) < \mu(X)$, whenever $\mu(X) > 0$, where $\mu$ can be replaced by some axiomatic measure of weak noncompactness.

Acknowledgment: I'm grateful to the referee’s valuable suggestions.

REFERENCES