SOLUTIONS AND POSITIVE SOLUTIONS TO SEMIPOSITONE DIRICHLET BVPS ON TIME SCALES

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ABSTRACT. In this paper, we are concerned with the following Dirichlet boundary value problem on a time scale

\[
-u^\Delta(t) = g(t, u(t)), \quad t \in [0, T]_T,
\]

\[
u(0) = 0 = u(\sigma^2(T)),
\]

where \( g : [0, T]_T \times [-\sigma(T)\sigma^2(T)M, +\infty) \to [-M, +\infty) \) is continuous and \( M > 0 \) is a constant, which implies that this problem is semipositone. For an arbitrary positive integer \( n \), some existence results for \( n \) solutions and/or positive solutions are established by using the well-known Guo-Krasnosel'skii fixed point theorem. Our conditions imposed on \( g \) are local. An example is also included to illustrate the importance of the results obtained.

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1. INTRODUCTION

Let \( T \) be a time scale (arbitrary nonempty closed subset of the real numbers \( \mathbb{R} \)). For each interval \( I \) of \( \mathbb{R} \), we denote by \( I_T = I \cap T \). For more details on time scales, one can refer to [1, 3, 7, 8]. In this paper, we consider solutions and positive solutions to the nonlinear Dirichlet boundary value problem (BVP for short) on a time scale \( T \)

\[
\begin{cases}
-u^\Delta(t) = g(t, u(t)), & t \in [0, T]_T, \\
u(0) = 0 = u(\sigma^2(T)),
\end{cases}
\]

where \( T > 0 \) is fixed and \( 0, T \in T \). Here, the solution \( u \) of the BVP (1.1) is called positive if \( u(t) > 0, \ t \in (0, \sigma^2(T))_T \). Throughout this paper, we assume that \( g : [0, T]_T \times [-\sigma(T)\sigma^2(T)M, +\infty) \to [-M, +\infty) \) is continuous and \( M > 0 \) is a constant; this implies that the BVP (1.1) is semipositone.

The BVP (1.1) has been discussed extensively when \( M = 0 \) (i.e., positone problem); see [2, 4, 5, 10] and the references therein. Recently, by using fixed point index theory, we [12] established some existence criteria for at least one positive solution to the BVP (1.1) assuming \( M > 0 \) (i.e., semipositone problem) and global conditions
on \( g \) (that is to say, these conditions are concerned with the growth of \( g \) on its whole domain). This paper is a continuation of our study in \([12]\). Our results show that the BVP (1.1) has at least \( n \) solutions and/or positive solutions provided that the “heights” of \( g \) on some bounded sets of its domain are appropriate, i.e., such existence results do not concern the growth of \( g \) outside these bounded sets. In other words, our conditions imposed on \( g \) are local. Our main idea comes from \([9, 13, 14]\), and our main tool is the well-known Guo-Krasnosel’skii fixed point theorem, which we state here for the convenience of the reader.

**Theorem 1.1** ([6]). Let \( X \) be a Banach space and \( K \) be a cone in \( X \). Assume that \( \Omega_1 \) and \( \Omega_2 \) are bounded open subsets of \( X \) with \( 0 \in \Omega_1 \), \( \overline{\Omega_1} \subset \Omega_2 \), and let \( \Phi : K \cap (\Omega_2 \setminus \Omega_1) \to K \) be a completely continuous operator such that either

\[
(i) \quad \|\Phi u\| \leq \|u\|, \ \forall u \in K \cap \partial \Omega_1 \quad \text{and} \quad \|\Phi u\| \geq \|u\|, \ \forall u \in K \cap \partial \Omega_2,
\]

or

\[
(ii) \quad \|\Phi u\| \geq \|u\|, \ \forall u \in K \cap \partial \Omega_1 \quad \text{and} \quad \|\Phi u\| \leq \|u\|, \ \forall u \in K \cap \partial \Omega_2.
\]

Then \( \Phi \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

### 2. MAIN RESULTS

Let

\[
X = \{ u \mid u : [0, \sigma^2(T)]_T \to \mathbb{R} \text{ is continuous} \}
\]

be equipped with the norm

\[
\|u\| = \max_{t \in [0, \sigma^2(T)]_T} |u(t)|.
\]

Then, \( X \) is a Banach space.

Define

\[
K = \{ u \in X \mid u(t) \geq q(t) \|u\|, \ t \in [0, \sigma^2(T)]_T \},
\]

where \( q(t) = \frac{t(\sigma^2(T)-t)}{(\sigma^2(T))^2}, \ t \in [0, \sigma^2(T)]_T \). Then, it is easy to see that \( K \) is a cone of \( X \).

To obtain a solution of the BVP (1.1), we require a mapping whose kernel \( G(t,s) \) is the Green’s function of the BVP

\[
(2.1) \quad \begin{cases} 
-u^{\Delta\Delta}(t) = 0, \ t \in [0, T]_T, \\
u(0) = 0 = u(\sigma^2(T)).
\end{cases}
\]

It is known that [3]

\[
(2.2) \quad G(t,s) = \frac{1}{\sigma^2(T)} \begin{cases} 
t(\sigma^2(T) - \sigma(s)), \ t \leq s, \\
\sigma(s)(\sigma^2(T) - t), \ t \geq \sigma(s).
\end{cases}
\]

For \( G(t,s) \), we have the following simple but important lemma.
Lemma 2.1. For any \( t \in [0, \sigma^2(T)]_T \) and \( s \in [0, \sigma(T)]_T \),
\begin{equation}
0 \leq G(t, s) \leq \frac{t (\sigma^2(T) - t)}{\sigma^2(T)}.
\end{equation}

Lemma 2.2. Let \( p(t) \) be the solution of the BVP
\begin{equation}
\begin{cases}
-p^\Delta(t) = 1, & t \in [0, T]_T, \\
p(0) = 0 = p(\sigma^2(T)).
\end{cases}
\end{equation}
Then,
\begin{equation}
0 \leq p(t) \leq q(t)\sigma(T)\sigma^2(T), \ t \in [0, \sigma^2(T)]_T.
\end{equation}
In particular,
\begin{equation}
0 \leq p(t) \leq \sigma(T)\sigma^2(T), \ t \in [0, \sigma^2(T)]_T.
\end{equation}

Proof. Since \( p(t) \) is the solution of the BVP (2.4), we know that
\begin{equation}
p(t) = \int_0^{\sigma(T)} G(t, s) \Delta s, \ t \in [0, \sigma^2(T)]_T.
\end{equation}
In view of Lemma 2.1, we have
\begin{equation}
0 \leq p(t) = \int_0^{\sigma(T)} G(t, s) \Delta s \leq \frac{t (\sigma^2(T) - t) \sigma(T)}{\sigma^2(T)} = q(t)\sigma(T)\sigma^2(T), \ t \in [0, \sigma^2(T)]_T.
\end{equation}

Let \( u_0(t) = Mp(t) \), \( t \in [0, \sigma^2(T)]_T \). We consider the following BVP
\begin{equation}
\begin{cases}
-u^\Delta(t) = g(t, u(t) - u_0(t)) + M, & t \in [0, T]_T, \\
u(0) = 0 = u(\sigma^2(T)).
\end{cases}
\end{equation}
It is easy to verify that if \( u(t) \) is a solution of the BVP (2.7), then \( u(t) - u_0(t) \) is a solution of the BVP (1.1). So, we will focus our attention on the BVP (2.7).

Since the BVP (2.7) is equivalent to the integral equation
\begin{equation}
u(t) = \int_0^{\sigma(T)} G(t, s)[g(s, u(s) - u_0(s)) + M] \Delta s, \ t \in [0, \sigma^2(T)]_T,
\end{equation}
we define the operator \( \Phi : K \to X \) as follows
\begin{equation}(\Phi u)(t) = \int_0^{\sigma(T)} G(t, s)[g(s, u(s) - u_0(s)) + M] \Delta s, \ t \in [0, \sigma^2(T)]_T.
\end{equation}
Noticing that
\begin{equation}
-\sigma(T)\sigma^2(T)M \leq u(t) - u_0(t) < +\infty \text{ for } u \in K \text{ and } t \in [0, \sigma^2(T)]_T,
\end{equation}
we know that \( \Phi : K \to X \) is well-defined.

Lemma 2.3. \( \Phi : K \to K \) is completely continuous.
Proof. Let \( u \in K \). By the definition of \( \Phi \), we know that \( (\Phi u)(0) = 0 = (\Phi u)(\sigma^2(T)) \). So, there exists a \( t_0 \in (0, \sigma^2(T))_T \) such that \( \|\Phi u\| = (\Phi u)(t_0) \). Since

\[
G(t, s) = \begin{cases} 
\frac{t}{t_0}, & t, 
\frac{(\sigma^2(T) - \sigma(s))}{\sigma(s)(\sigma^2(T) - t_0)}, & t \leq s < t_0, \\
\frac{\sigma(s)(\sigma^2(T) - t)}{t_0(\sigma^2(T) - \sigma(s))}, & t_0 \leq s < t, \\
\frac{\sigma^2(T) - t}{\sigma(T) - t_0}, & t, 
\end{cases}
\]

we obtain that

\[
(2.11) \quad \frac{G(t, s)}{G(t_0, s)} \geq q(t), \quad t \in [0, \sigma^2(T)]_T \text{ and } s \in [0, \sigma(T)]_T.
\]

So,

\[
(\Phi u)(t) = \int_0^{\sigma(T)} G(t, s)[g(s, u(s) - u_0(s)) + M] \Delta s
\]

\[
= \int_0^{\sigma(T)} \frac{G(t, s)}{G(t_0, s)} G(t_0, s)[g(s, u(s) - u_0(s)) + M] \Delta s
\]

\[
\geq q(t) \int_0^{\sigma(T)} G(t_0, s)[g(s, u(s) - u_0(s)) + M] \Delta s
\]

\[
= q(t)(\Phi u)(t_0)
\]

\[
= q(t)\|\Phi u\|, \quad t \in [0, \sigma^2(T)]_T,
\]

which shows that \( \Phi u \in K \). Furthermore, by using similar arguments to those in [11], we can prove that \( \Phi : K \to K \) is completely continuous. \( \square \)

In the remainder of this paper, we let \( \xi, \eta \in T \) be such that \( 0 < \xi < \eta < T \) and denote

\[
A = \left[ \max_{t \in [0, \sigma^2(T)]_T} \int_0^{\sigma(T)} G(t, s) \Delta s \right]^{-1},
\]

\[
B = \left[ \max_{t \in [0, \sigma^2(T)]_T} \int_{\xi}^{\eta} G(t, s) \Delta s \right]^{-1},
\]

\[
\varphi(r) = \max \{ g(t, u) + M \mid t \in [0, T]_T, \ u \in [-\sigma(T)\sigma^2(T)M, r] \}
\]

and

\[
\psi(r) = \min \left\{ g(t, u) + M \mid t \in [\xi, \eta]_T, \ u \in \left[ \frac{\xi(\sigma^2(T) - \eta)r}{(\sigma^2(T))^2} - \sigma(T)\sigma^2(T)M, r \right] \right\}.
\]

It is obvious that \( 0 < A < B \).

Now, we state and prove a basic existence criterion as follows:
Theorem 2.4. Assume that there exist two positive numbers \( r_1 \) and \( r_2 \) such that \( \varphi(r_1) \leq r_1A \) and \( \psi(r_2) \geq r_2B \). Then, the BVP \((1.1)\) has at least one solution \( u^* \) satisfying \( u^* + u_0 \in K \) and

\[
\min \{r_1, r_2\} \leq \|u^* + u_0\| \leq \max \{r_1, r_2\}.
\]

Moreover, if \( \min \{r_1, r_2\} > \sigma(T)\sigma^2(T)M \), then \( u^* \) is a positive solution of the BVP \((1.1)\).

Proof. Since \( 0 < A < B \), it is easy to see that \( r_1 \neq r_2 \). Without loss of generality, we assume that \( r_1 < r_2 \). Let

\[
\Omega_i = \{u \in \mathbb{X} \mid \|u\| < r_i\}, \quad i = 1, 2.
\]

If \( u \in K \cap \partial\Omega_1 \), i.e., \( u \in K \) and \( \|u\| = r_1 \), then \( 0 \leq u(t) \leq r_1 \), \( t \in [0, \sigma^2(T)] \). So,

\[
-\sigma(T)\sigma^2(T)M \leq u(t) - u_0(t) \leq r_1, \quad t \in [0, \sigma^2(T)] \in T.
\]

And so,

\[
(2.12) \quad g(t, u(t) - u_0(t)) + M \leq \varphi(r_1) \leq r_1A, \quad t \in [0, T] \in T.
\]

It follows that

\[
(\Phi u)(t) = \int_0^{\sigma(T)} G(t, s)[g(s, u(s) - u_0(s)) + M] \Delta s
\]

\[
\leq r_1 A \int_0^{\sigma(T)} G(t, s) \Delta s
\]

\[
\leq r_1 A \max_{t \in [0, \sigma^2(T)]} \int_0^{\sigma(T)} G(t, s) \Delta s
\]

\[
= r_1, \quad t \in [0, \sigma^2(T)] \in T,
\]

which shows that

\[
(2.13) \quad \|\Phi u\| \leq \|u\| \quad \text{for} \quad u \in K \cap \partial\Omega_1.
\]

If \( u \in K \cap \partial\Omega_2 \), i.e., \( u \in K \) and \( \|u\| = r_2 \), then for \( t \in [\xi, \eta] \), we have

\[
\frac{\xi (\sigma^2(T) - \eta) r_2}{(\sigma^2(T))^2} \leq q(t) r_2 \leq u(t) \leq r_2
\]

and

\[
\frac{\xi (\sigma^2(T) - \eta) r_2}{(\sigma^2(T))^2} - \sigma(T)\sigma^2(T)M \leq u(t) - u_0(t) \leq r_2.
\]

So,

\[
(2.14) \quad g(t, u(t) - u_0(t)) + M \geq \psi(r_2) \geq r_2 B, \quad t \in [\xi, \eta] \in T.
\]
It follows that
\[
\|\Phi u\| = \max_{t \in [0, \sigma^2(T)]} \int_0^{\sigma(T)} G(t, s)[g(s, u(s) - u_0(s))] + M \Delta s \\
\geq \max_{t \in [0, \sigma^2(T)]} \int_0^\eta G(t, s)[g(s, u(s) - u_0(s))] + M \Delta s \\
\geq r_2 B \max_{t \in [0, \sigma^2(T)]} \int_0^\eta G(t, s) \Delta s \\
= r_2,
\]
i.e.,
\[
(2.15) \quad \|\Phi u\| \geq \|u\| \text{ for } u \in K \cap \partial \Omega_2.
\]

In view of (2.13), (2.15), Lemma 2.3, and Theorem 1.1, we know that the operator \( \Phi \) has at least one fixed point \( u \in K \cap \overline{(\Omega_2 \setminus \Omega_1)} \), which implies that the BVP (2.7) has at least one solution \( u \in K \) such that \( r_1 \leq \|u\| \leq r_2 \). Therefore, \( u^* = u - u_0 \) is a solution of the BVP (1.1) such that
\[
(2.16) \quad u^* + u_0 \in K \text{ and } r_1 \leq \|u^* + u_0\| \leq r_2.
\]

Moreover, if \( r_1 > \sigma(T) \sigma^2(T) M \), then for any \( t \in (0, \sigma^2(T)) \), by (2.16) and Lemma 2.2, we have
\[
\begin{align*}
 u^*(t) &= [u^*(t) + u_0(t)] - u_0(t) = [u^*(t) + u_0(t)] - M p(t) \\
 &\geq q(t) \|u^* + u_0\| - q(t) \sigma(T) \sigma^2(T) M \\
 &\geq q(t) r_1 - q(t) \sigma(T) \sigma^2(T) M \\
 &= [r_1 - \sigma(T) \sigma^2(T) M] q(t) \\
 &> 0,
\end{align*}
\]
which shows that \( u^* \) is a positive solution of the BVP (1.1).

Next, based on Theorem 2.4, we establish some criteria which ensure the existence of \( n \) solutions and/or positive solutions to the BVP (1.1); here \( n \) is an arbitrary positive integer.

**Corollary 2.5.** Suppose that there exist three positive numbers \( r_1, r_2 \) and \( r_3 \) with \( r_1 < r_2 < r_3 \) such that one of the following conditions is satisfied:

\[(a) \ \varphi(r_1) \leq r_1 A, \ \psi(r_2) > r_2 B, \ \varphi(r_3) \leq r_3 A, \]
or

\[(b) \ \psi(r_1) \geq r_1 B, \ \varphi(r_2) < r_2 A, \ \psi(r_3) \geq r_3 B. \]
Then the BVP (1.1) has at least two solutions \( u_1^*, u_2^* \) satisfying \( u_1^* + u_0, u_2^* + u_0 \in K \) and

\[
r_1 \leq \|u_1^* + u_0\| < r_2 < \|u_2^* + u_0\| \leq r_3.
\]

Moreover, if \( r_2 > \sigma(T)\sigma^2(T)M \), then \( u_2^* \) is a positive solution of the BVP (1.1), and if \( r_1 > \sigma(T)\sigma^2(T)M \), then \( u_1^*, u_2^* \) are both positive solutions of the BVP (1.1).

**Proof.** It is enough to prove case (a). Since \( \frac{\psi(r)}{r^2} : (0, +\infty) \to [0, +\infty) \) is continuous and \( \frac{\psi(r_2)}{r_2} > B \), there exist two positive numbers \( \tilde{r}_2 \) and \( r_2 \) with \( r_1 < \tilde{r}_2 < r_2 < r_3 < r_3 \) such that \( \psi(\tilde{r}_2) \geq \tilde{r}_2 B \) and \( \psi(r_2) \geq r_2 B \). It follows from Theorem 2.4 that the BVP (1.1) has at least two solutions \( u_1^*, u_2^* \) satisfying \( u_1^* + u_0, u_2^* + u_0 \in K \) and

\[
r_1 \leq \|u_1^* + u_0\| \leq \tilde{r}_2 < r_2 < r_3 \leq \|u_2^* + u_0\| \leq r_3.
\]

\[\square\]

**Corollary 2.6.** Suppose that there exist four positive numbers \( r_1, r_2, r_3 \) and \( r_4 \) with \( r_1 < r_2 < r_3 < r_4 \) such that one of the following conditions is satisfied:

(a) \( \varphi(r_1) \leq r_1 A, \varphi(r_2) > r_2 B, \varphi(r_3) < r_3 A, \varphi(r_4) \geq r_4 B \),

or

(b) \( \psi(r_1) \geq r_1 B, \varphi(r_2) < r_2 A, \psi(r_3) > r_3 B, \varphi(r_4) \leq r_4 A \).

Then the BVP (1.1) has at least three solutions \( u_1^*, u_2^*, u_3^* \) satisfying \( u_1^* + u_0, u_2^* + u_0, u_3^* + u_0 \in K \) and

\[
r_1 \leq \|u_1^* + u_0\| < r_2 < \|u_2^* + u_0\| < r_3 < \|u_3^* + u_0\| \leq r_4.
\]

Moreover, if \( r_3 > \sigma(T)\sigma^2(T)M \), then \( u_3^* \) is a positive solution of the BVP (1.1), if \( r_2 > \sigma(T)\sigma^2(T)M \), then \( u_2^*, u_3^* \) are both positive solutions of the BVP (1.1), and if \( r_1 > \sigma(T)\sigma^2(T)M \), then \( u_1^*, u_2^*, u_3^* \) are all positive solutions of the BVP (1.1).

**Proof.** We only prove case (a). Since \( \frac{\psi(r)}{r^2} : (0, +\infty) \to [0, +\infty), \frac{\varphi(r)}{r^2} : (0, +\infty) \to [0, +\infty) \) are continuous and \( \frac{\psi(r_2)}{r_2} > B, \frac{\varphi(r_3)}{r_3} < A \), there exist four positive numbers \( \tilde{r}_2, \tilde{r}_3, \tilde{r}_3, \tilde{r}_3 \) with \( r_1 < \tilde{r}_2 < r_2 < \tilde{r}_3 < \tilde{r}_3 < \tilde{r}_3 < r_3 \) such that \( \psi(\tilde{r}_2) \geq \tilde{r}_2 B \), \( \psi(\tilde{r}_3) \geq \tilde{r}_3 B \), \( \varphi(\tilde{r}_3) \leq \tilde{r}_3 A \), \( \varphi(\tilde{r}_3) \geq \tilde{r}_3 A \). It follows from Theorem 2.4 that the BVP (1.1) has at least three solutions \( u_1^*, u_2^*, u_3^* \) satisfying \( u_1^* + u_0, u_2^* + u_0, u_3^* + u_0 \in K \) and

\[
r_1 \leq \|u_1^* + u_0\| \leq \tilde{r}_2 < r_2 < \tilde{r}_3 \leq \|u_2^* + u_0\| \leq \tilde{r}_3 < r_3 \leq \|u_3^* + u_0\| \leq r_4.
\]

\[\square\]

Similarly, for arbitrary positive integer \( n \), the existence results of \( n \) solutions and/or positive solutions to the BVP (1.1) still hold.
Example 2.7. Consider the following BVP

\begin{equation}
\begin{cases}
-u^\Delta(t) = 128\sqrt{t(u(t) + 1)} - 1, & t \in [0, 1], \\
u(0) = 0 = u(1),
\end{cases}
\end{equation}

where \( T = \{0, \frac{1}{4}\} \cup \left[\frac{1}{4}, 1\right]. \)

Let \( T = 1, \) \( \xi = \frac{1}{4} \) and \( \eta = \frac{1}{2}. \) We first compute the values of \( A \) and \( B. \) In view of

\[
\int_0^\frac{1}{2} G(t, s) \Delta s = \sum_{s \in [0, \frac{1}{2}) T} \mu(s) G(t, s) = \begin{cases}
0, & t = 0, \\
\frac{5}{64}, & t = \frac{1}{4}, \\
\frac{1}{16}, & t \geq \frac{1}{2},
\end{cases}
\]

and

\[
\int_{\frac{1}{4}}^1 G(t, s) \Delta s = \begin{cases}
\frac{t^2}{2} + \frac{5t}{8} - \frac{1}{8}, & t \leq \frac{1}{2}, \\
\frac{1}{2} - \frac{t^2}{2} + \frac{7}{16}, & t \geq \frac{1}{2},
\end{cases}
\]

we have

\[
\int_0^1 G(t, s) \Delta s = \begin{cases}
0, & t = 0, \\
\frac{7}{64}, & t = \frac{1}{4}, \\
\frac{1}{2} - \frac{t^2}{2} + \frac{7}{16}, & t \geq \frac{1}{2},
\end{cases}
\]

So,

\[
A = \left[ \max_{t \in [0, 1]} \int_0^1 G(t, s) \Delta s \right]^{-1} = \frac{32}{5},
\]

Since

\[
\int_{\frac{1}{4}}^{\frac{1}{2}} G(t, s) \Delta s = \sum_{s \in [\frac{1}{4}, \frac{1}{2}) T} \mu(s) G(t, s) = \begin{cases}
\frac{7}{8}, & t \leq \frac{1}{4}, \\
\frac{1}{8}, & t \geq \frac{1}{2},
\end{cases}
\]

we get

\[
B = \left[ \max_{t \in [0, 1]} \int_{\frac{1}{4}}^{\frac{1}{2}} G(t, s) \Delta s \right]^{-1} = 16.
\]

Then, it is easy to verify that all the conditions of Theorem 2.4 are satisfied if we let \( g(t, u) = 128\sqrt{t(u + 1)} - 1, (t, u) \in [0, 1] \times [-1, +\infty), M = 1, r_1 = 10^4 \) and \( r_2 = 2. \) So, the BVP (2.17) has at least one positive solution.

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