QUENCHING CRITERIA FOR A DEGENERATE PARABOLIC PROBLEM DUE TO A CONCENTRATED NONLINEAR SOURCE

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ABSTRACT. A criterion for the quenching of the solution for a degenerate semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source situated at \( b \) is given. The locations of \( b \) for global existence of the solution and for the quenching of the solution are given.

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1. INTRODUCTION

Let \( q, a, T \) and \( b \) be any numbers such that \( q \geq 0, a > 0, T > 0, \) and \( 0 < b < 1. \) Also, let \( D \) denote the interval \((0,1)\), and \( \overline{D} \) be its closure. We consider the following degenerate semilinear parabolic first initial-boundary value problem with a concentrated nonlinear source situated at \( b \),

\[
\begin{cases}
x^q u_t - u_{xx} = a\delta(x-b)f(u(x,t)) & \text{in } D \times (0,T], \\
u(x,0) = 0 & \text{on } \overline{D}, \\
u(0,t) = u(1,t) = 0 & \text{for } 0 < t \leq T,
\end{cases}
\]

(1.1)

where \( \delta(x) \) is the Dirac delta function, \( f \) is a given function such that \( \lim_{u \to c^-} f(u) = \infty \) for some positive constant \( c \), and \( f(u) \) and its derivatives \( f'(u) \) and \( f''(u) \) are positive for \( 0 \leq u < c \). The case \( q = 0 \) was studied by Deng and Roberts [7] by analyzing its corresponding nonlinear Volterra equation at the site \( b \) of the concentrated source. Instead of studying a solution \( u(b,t) \) of the nonlinear Volterra equation, we would like to investigate a solution \( u(x,t) \) of the degenerate problem (1.1).

The right-hand side of the partial differential equation in (1.1) has the term \( \delta(x-b) \). This implies that \( u_x \) has a jump discontinuity at \( x = b \). Thus, a solution of the problem (1.1) is a continuous function satisfying (1.1). In the proof of Theorem 3 of Chan and Jiang [4], it is shown that \( u_{xx} \geq 0 \) for \( x \in (0,b) \) and \( x \in (b,1) \). It follows from the differential equation in (1.1) that \( u_t(b,t) = \infty \) for each \( t > 0 \). Hence, we say that a solution \( u \) of the problem (1.1) is said to quench if there exists some \( t_q \) such that

\[
\max\{u(x,t) : x \in \overline{D}\} \to c^- \text{ as } t \to t_q
\]
(cf. Chan and Liu [5]). If \( t_q \) is finite, then \( u \) is said to quench in a finite time. On the other hand, if \( t_q = \infty \), then \( u \) is said to quench in infinite time.

Let \( G (x,t;\xi,\tau) \) denote Green’s function corresponding to the problem (1.1), and \( t_q \) denote the supremum of all \( t_1 \) such that on \([0,t_1]\), the integral equation,

\[
(1.2) \quad u (x,t) = a \int_0^t G (x,t;b,\tau) f (u (b,\tau)) d\tau,
\]
corresponding to the problem (1.1) has a unique nonnegative continuous solution. For ease of reference, we summarize the main results of Theorems 1, 2 and 3 of Chan and Jiang [4] as Theorem 1.1 below.

**Theorem 1.1.** There exists some \( t_q (\leq \infty) \) such that for \( 0 \leq t < t_q \), the integral equation (1.2) has a unique nonnegative continuous solution \( u (x,t) \), which is a strictly increasing function of \( t \) in \( D \). Before a quenching occurs, \( u \) is the solution of the problem (1.1), and attains its maximum at \((b,t)\) for each \( t > 0 \). If \( t_q \) is finite, then \( u \) quenches at \( t_q \). Furthermore, if \( u \) quenches, then \( b \) is the single quenching point.

In Section 2, we give a criterion for the quenching. It turns out that the forcing term \( f (u) \) need not be superlinear in \( u \) for a quenching to occur. This is in sharp contrast with the blow-up phenomenon, which requires the forcing term to be superlinear (cf. Chan and Tian [6]). In Section 3, we find the exact position \( b^* \) for the problem (1.1) such that \( u \) never quenches for \( b \in (0,b^*] \cup [1-b^*,1) \), and \( u \) always quenches in a finite time for \( b \in (b^*,1-b^*) \). For illustration, an example is given.

### 2. A QUENCHING CRITERION

Let

\[
\mu (t) = \int_D x^q \phi (x) u (x,t) dx,
\]
where \( \phi \) denotes the normalized fundamental eigenfunction of the problem,

\[
\phi'' + \lambda x^q \phi = 0, \quad \phi(0) = \phi(1) = 0,
\]
with \( \lambda \) denoting its corresponding eigenvalue, which is positive (cf. Chan and Chan [2]). Below is a quenching criterion.

**Theorem 2.1.** If there exist constants \( c_1 (>0) \) and \( c_2 (\geq 0) \) such that

\[
(2.1) \quad \sqrt{1+q\phi (b)} f (u (b,t)) \geq c_1 + c_2 u (b,t),
\]

\[
(2.2) \quad \frac{\lambda}{a} > c_2, \quad \frac{ac_1}{\lambda - ac_2} > c,
\]
then \( u \) quenches in a finite time. Furthermore, an upper bound for the quenching time is given by

\[
\frac{1}{\lambda - ac_2} \ln \left[ 1 - \frac{(\lambda - ac_2)c}{ac_1} \right]^{-1}.
\]
Proof. Multiplying the partial differential equation in (1.1) by \( \phi \), and integrating with respect to \( x \) over \( D \), we obtain

\[
\mu' (t) + \lambda \mu (t) = a \phi (b) f (u (b,t)) .
\]

Since \( u (x,t) \leq u (b,t) \), we have

\[
\mu (t) \leq \left( \int_D x^q \phi (x) \, dx \right) u (b,t) .
\]

It follows from the Schwarz inequality and \( \int_D x^q \phi^2 (x) \, dx = 1 \) that

\[
\mu (t) \leq \left( \int_D x^q \phi^2 (x) \, dx \right)^{1/2} \left( \int_D x^q \, dx \right)^{1/2} u (b,t) \]

\[
= \frac{1}{\sqrt{1+q}} u (b,t) .
\]

By (2.1),

\[
a \phi (b) f (u (b,t)) \geq \frac{a}{\sqrt{1+q}} \left( c_1 + c_2 u (b,t) \right) \geq a \left( \frac{1}{\sqrt{1+q}} c_1 + c_2 \mu (t) \right) .
\]

From (2.3),

\[
\mu' (t) + (\lambda - ac_2) \mu (t) \geq a \frac{c_1}{\sqrt{1+q}} c_1.
\]

Since \( \mu (0) = 0 \), we obtain

\[
\mu (t) \geq \frac{ac_1}{\sqrt{1+q} (\lambda - ac_2)} \left[ 1 - e^{-(\lambda - ac_2)t} \right] .
\]

Hence,

\[
u (b,t) \geq \sqrt{1+q} \mu (t) \geq \frac{ac_1}{\lambda - ac_2} \left[ 1 - e^{-(\lambda - ac_2)t} \right] .
\]

From (2.2), there exists some finite \( t_q \) such that \( u \) quenches at \((b, t_q)\). An upper bound for the quenching time follows by setting the right-hand side of the above inequalities equal to \( c \) to evaluate \( t \).

\[
\square
\]

3. CRITICAL POSITION \( b^* \)

Let \( \lim_{t \to \infty} u (x,t) \) be denoted by \( U (x) \). For ease of reference, let us summarize the main results of Section 3 of Chan and Jiang [4] in the following theorem.

**Theorem 3.1.** There exists a critical length \( a^* \) such that \( u \) exists on \( \bar{D} \) for all \( t > 0 \) if \( a \leq a^* \), and \( u \) quenches in a finite time if \( a > a^* \). The critical length \( a^* \) is determined as the supremum of all positive values \( a \) for which a solution \( U \) of the nonlinear two-point boundary value problem,

\[
-U'' (x) = a \delta (x-b) f (U(x)) \text{ in } D, \quad U(0) = U(1) = 0,
\]

exists.
exists. Furthermore, \( u(x, t) < U(x) \) in \( D \times (0, \infty) \),

\[
U(x) = ag(x; b) f(U(b)),
\]

where

\[
g(x; \xi) = \begin{cases} 
\xi (1 - x), & 0 \leq \xi \leq x, \\
x (1 - \xi), & x < \xi \leq 1,
\end{cases}
\]

is Green’s function corresponding to the problem (3.1),

\[
a^* = \frac{1}{b(1 - b)} \max_{0 \leq s \leq c} \left( \frac{s}{f(s)} \right) \text{ for a given } b \in D.
\]

As a consequence of the above theorem, the solution \( u \) does not quench in infinite time. We note from (3.3) that \( a^* \) depends on \( b \). For a given \( a (> a^*) \), there exists a position \( b \) such that the problem (1.1) quenches in a finite time. Chan and Boonklurb [1] studied the critical position of the concentrated source for a blow-up problem. Here, we give an analogous argument for the quenching problem (1.1). To find a position \( b \) for the same given \( a (> a^*) \) such that the solution \( u \) exists for all \( t > 0 \), let us first consider the problem (1.1) with \( q = 0 \), namely,

\[
\begin{align*}
vt - v_{xx} &= a\delta(x - b)f(v(x, t)) \quad \text{in } D \times (0, T], \\
v(x, 0) &= 0 \text{ on } \bar{D},
\end{align*}
\]

From Theorem 1.1, the quenching set is the single point \( x = b \), and

\[
v(b, t) = a \int_0^t G_0(b, t; b, \tau)f(v(b, \tau))d\tau,
\]

where

\[
G_0(x, t; \xi, \tau) = 2 \sum_{n=1}^{\infty} (\sin n\pi x)(\sin n\pi \xi)e^{-n^2\pi^2(t - \tau)} \quad \text{for } t > \tau
\]

is Green’s function corresponding to the problem (3.4). From Olmstead and Roberts [9],

\[
\int_0^t G_0(b, t; b, \tau)d\tau = b(1 - b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n\pi b}{n^2} e^{-n^2\pi^2 t}.
\]

Since \( \sum_{n=1}^{\infty} (\sin^2 n\pi b)e^{-n^2\pi^2 t}/n^2 \) and \( 2 \sum_{n=1}^{\infty} (\sin^2 n\pi b)e^{-n^2\pi^2 t} \) converge uniformly in \((0, t)\), we have

\[
\frac{\partial}{\partial t} \left( \int_0^t G_0(b, t; b, \tau)d\tau \right) = 2 \sum_{n=1}^{\infty} (\sin^2 n\pi b)e^{-n^2\pi^2 t} > 0,
\]

(3.6)

\[
\lim_{t \to \infty} \int_0^t G_0(b, t; b, \tau)d\tau = b(1 - b) - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 n\pi b}{n^2} \lim_{t \to \infty} e^{-n^2\pi^2 t} = b(1 - b).
\]
From Theorem 1.1, \( v(x,t) \) attains its maximum \( M \) at \((b,\theta)\) for \( 0 \leq t \leq \theta \). Thus given any positive number \( M(<c) \), it follows from (3.5) and (3.6) that for \( 0 \leq t \leq \theta \),

\[
v(b,t) \leq a f(M) \int_0^t G_0(b,\tau;b,\tau) d\tau \leq a f(M) b(1-b).
\]

In order that \( a f(M) b(1-b) \leq M \) so that \( v \) exists for all \( t > 0 \), we choose \( b \) in such a way that

\[
0 < b \leq \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4M}{af(M)}} \right) \quad \text{or} \quad \frac{1}{2} \left( 1 + \sqrt{1 - \frac{4M}{af(M)}} \right) \leq b < 1.
\]

Since \( v \) is a nondecreasing function of \( t \), we have for \( 0 \leq x \leq 1 \) and \( q > 0 \),

\[
x^q v_t - v_{xx} \leq v_t - v_{xx},
\]

which implies that the solution of the problem (1.1) is a lower solution of the problem (3.4). Thus under the above condition (3.7) on \( b \), the solution of (1.1) exists for all \( t > 0 \).

Let us consider the function

\[
\psi(U(b)) = \frac{U(b)}{f(U(b))}.
\]

Since \( \psi(U(b)) > 0 \) for \( 0 < U(b) < c \), and \( \psi(0) = 0 = \lim_{U(b) \to c^-} \psi(U(b)) \), a direct computation shows that \( \psi(U(b)) \) attains its maximum when \( \psi(U(b)) = 1/f'(U(b)) \), where \( U(b) \in (0,c) \) by Rolle’s Theorem. Thus, \( \max(U(b)/f(U(b))) \) occurs when

\[
\frac{U(b)}{f(U(b))} = \frac{1}{f'(U(b))}, \quad \text{where} \ 0 < U(b) < c.
\]

This also implies that \( U(x) \) exists when \( a = a^* \).

From (3.2), \( U(b) = ab(1-b)f(U(b)) \). We would like to know how \( U(b) \) behaves as \( b \) varies when \( a > a^* \). A direct calculation gives

\[
U'(b) = \frac{a(1-2b)f(U(b))}{1-ab(1-b)f'(U(b))}.
\]

Since \( a > a^* \), and \( 1/4 \geq b(1-b) \), we have

\[
1 - \frac{4}{\max_{0 \leq U \leq c} U(b)} U(b) > 1 - \frac{4}{a^* \max_{0 \leq U \leq c} U(b)} U(b) \geq 0.
\]

Thus for

\[
b \in \left( 0, \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{\max_{0 \leq U \leq c} U(b)}} \frac{U(b)}{f(U(b))} \right) \right),
\]

the numerator is positive. Also,

\[
b < \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{\max_{0 \leq U \leq c} U(b)}} \frac{U(b)}{f(U(b))} \right)
\]
gives
\[ b - \frac{1}{2} < -\frac{1}{2} \sqrt{1 - \frac{4}{a} \max_{0 \leq U \leq c} f(U(b))} < 0. \]

We have
\[ \left( b - \frac{1}{2} \right)^2 > \frac{1}{4} \left( 1 - \frac{4}{a} \max_{0 \leq U \leq c} f(U(b)) \right), \]
which by (3.8) gives
\[ 1 - ab (1 - b) f'(U(b)) > 0, \]
and hence, \( U'(b) > 0 \). Thus for a given \( a > a^* \), the function \( U(b) \) is a strictly increasing function of \( b \) for
\[ b \in \left( 0, 1 \right), \]
Similarly for a given \( a > a^* \), the function \( U(b) \) is a strictly decreasing function of \( b \) for
\[ b \in \left( \frac{1}{2}, 1 \right). \]

Hence on the interval \((0, 1/2)\), the position \( b \) for global existence of \( u \) is closer to 0 than the position \( b \) for the quenching of \( u \) in a finite time. On the other hand, on the interval \((1/2, 1)\), the position \( b \) for global existence of \( u \) is closer to 1 than the position \( b \) for the quenching of \( u \) in a finite time. Thus, there exists \( b^* \in (0, 1/2) \) such that the steady state \( U(x) \) exists for \( b \in (0, b^*) \cup (1 - b^*, 1) \), and does not exist for \( b \in (b^*, 1 - b^*) \). We note that
\[ b^* = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{a} \max_{0 \leq U \leq c} f(U(b))} \right), \]
and is attained for \( 0 < U(b) < c \) by (3.8). Since \( u(x, t) \leq U(x) = \lim_{t \to \infty} u(x, t) \) in \( D \times (0, \infty) \) when \( U \) exists, we have for \( b \in (0, b^*) \cup (1 - b^*, 1) \), \( u \) exists for \( 0 \leq t < \infty \), and for \( b \in (b^*, 1 - b^*) \), \( u \) quenches in a finite time.

The above discussion gives the following result.

**Theorem 3.2.** For \( a > a^* \), the solution of the problem (1.1) exists globally for \( b \in (0, b^*) \cup (1 - b^*, 1) \), and quenches in a finite time for \( b \in (b^*, 1 - b^*) \).

For illustration, let \( f(u) = (1 - u)^{-p} \). A direct computation shows that
\[ a^* = \frac{p}{b(1 - b)(1 + p)^{1+p}}, \]
\[ b^* = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4p^2}{a(1 + p)^{1+p}}} \right). \]

When \( p = 1 \) and \( b = 1/2 \), we have \( a^* = 1 \), and \( b^* = (1 - \sqrt{1 - a^{-1}}) / 2 \) for \( a > 1 \). We note that the concept of the quenching was introduced by Kawarada [8] through the
following problem, which arises in the study of a polarization phenomenon in ionic conductors:

\[ u_t - u_{xx} = \frac{1}{1 - u} \quad \text{in} \quad (0, a) \times (0, T), \]

\[ u(x, 0) = 0 \quad \text{on} \quad 0 \leq x \leq a, \]

\[ u(0, t) = u(a, t) = 0 \quad \text{for} \quad 0 < t \leq T. \]

Its \( a^* = 1.5303 \) (to five significant figures) (cf. Chan and Chen [3]). Thus, the presence of the concentrated source shortens the critical length.

REFERENCES


