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SECOND ORDER CARATHEODORY AND DISCONTINUOUS INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH ALGEBRAS

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ABSTRACT. In this paper an existence theorem for the second order integro-differential equations in Banach algebras is proved under the mixed generalized Lipschitz and Carathéodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions.

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1. INTRODUCTION

The study of nonlinear differential and integral equations in Banach algebras is initiated in the works of Dhage [1] and Dhage and O'Regan [7] where the existence theory for first order ordinary differential equations is discussed. The main tools used in the study is hybrid fixed point theorems of Dhage type [4]. Like nonlinear perturbed differential equations, the solutions of such differential equations are also obtained under the mixed Lipschitz and compactness type conditions. In recent years, the topic of nonlinear differential equations in Banach algebras is received the attention of several authors and at present, considerable literature available is this direction. In this paper, we deal with the second order ordinary integro-differential equations in Banach algebras and discuss the existence results under mixed Lipschitz and Carathéodory conditions. We will employ the hybrid fixed point theorems of Dhage [2, 3, 5] for proving our main existence results. We claim that the nonlinear differential equation as well as the existence results of this are new to the literature on the theory of ordinary integro-differential equations.

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the second order integro-differential equation (in short IGDE)

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\Big[\frac{x(t)}{f(t,x(t))}\Big] = \int_0^t g(s,x(s))\,ds \ \text{ a.e. } \ t \in J \\ &x(0) = x_0,\, x'(0) = x_1, \end{aligned}\right\} \tag{1.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $g : I \times \mathbb{R} \to \mathbb{R}$.

By a solution of IGDE (1.1) we mean a function $x \in AC^2(J, \mathbb{R})$ that satisfies

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(i) the function $t \mapsto \left(\dfrac{x}{f(t,x)}\right)$ is derivable and the derivative $\left(\dfrac{x}{f(t,x)}\right)'$ is absolutely continuous on J for all $x \in \mathbb{R}$, and

(ii) x satisfies the equations in (1.1),

where $AC^2(J,\mathbb{R})$ is the space of continuous functions whose second derivative exists and is absolutely continuous real-valued functions on J.

Initial value problems for first order integro-differential equations was studied recently in [6]. Here we extend the results to initial value problems for second order integro-differential equations.

Our method of study is to convert the IGDE (1.1) into an equivalent integral equation and apply the fixed point theorems of Dhage [2, 3, 5] under suitable conditions on the nonlinearities f and g. In the following section we shall give some preliminaries needed in the sequel. The main existence result is proved in Section 3. In Section 4 we study the existence of extremal solutions under monotonicity conditions.

Let X be a Banach algebra with norm $\|\cdot\|$. A mapping $A : X \to X$ is called **$\mathcal{D}$-Lipschitz** if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \tag{1.2}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case when $\psi(r) = \alpha r$ $(\alpha > 0)$, A is called a **$\mathcal{D}$-Lipchitz** with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a **$\mathcal{D}$-contraction** with a contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear **$\mathcal{D}$-contraction** on X. Sometimes we call the function ψ a **$\mathcal{D}$-function** (for convenience) for the **$\mathcal{D}$-Lipschitz** mappings T on X.

An operator $T : X \to X$ is called **compact** if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly $T : X \to X$ is called **totally bounded** if T maps a bounded subset of X into the relatively compact subset of X. Finally $T : X \to X$ is called **completely continuous** operator if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [4] is embodied in the following theorem.

Theorem 1.1. (Dhage [4]) *Let X be a Banach algebra and let $A, B : X \to X$ be two operators satisfying*

(a) *A is Lipschitz with a Lipschitz constant α,*

(b) *B is compact and continuous, and*

(c) *$\alpha M < 1$, where $M = \|B(X)\| := \sup\{\|Bx\| : x \in X\}$.*

Then either

(i) *the equation $\lambda[Ax\, Bx] = x$ has a solution for $\lambda = 1$, or*

(ii) *the set* $\mathcal{E} = \{u \in X \mid \lambda[Au\,Bx] = u,\, 0 < \lambda < 1\}$ *is unbounded.*

A non-empty closed set K in a Banach algebra X is called a **cone** if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called to be **positive** if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called to be **normal** if the norm $\|\cdot\|$ is monotone increasing on K. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [9].

Lemma 1.2. (Dhage [3]). Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.

For any $a, b \in X, a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

We use the following fixed point theorem of Dhage [2, 3, 5] for proving the existence of extremal solutions for the IGDE (1.1) under certain monotonicity conditions.

Theorem 1.3. (Dhage [2]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*
(b) *B is completely continuous,*
(c) *$Ax\,Bx \in [a, b]$ for each $x \in [a, b]$, and*
(d) *A and B are nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 1.4. (Dhage [5]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is completely continuous,*
(b) *B is totally bounded,*
(c) *$Ax\,By \in [a, b]$ for all $x, y \in [a, b]$, and*
(d) *A and B is nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a, b]$.

Theorem 1.5. (Dhage [5]). *Let K be a cone in a Banach algebra X and let $a, b \in X$. Suppose that $A, B : [a, b] \to K$ are two operators such that*

(a) *A is Lipschitz with a Lipschitz constant α,*
(b) *B is totally bounded,*
(c) *$Ax\,By \in [a,b]$ for each $x, y \in [a,b]$, and*
(d) *A and B is nondecreasing.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx = x$ has a least and a greatest positive solution in $[a,b]$, whenever $\alpha M < 1$, where $M = \|B([a,b])\| := \sup\{\|Bx\| : x \in [a,b]\}$.

Remark 1.6. Note that hypothesis (c) of Theorems 1.3, 1.4, and 1.5 holds if the operators A and B are positive monotone increasing and there exist elements a and b in X such that $a \leq Aa\,Ba$ and $Ab\,Bb \leq b$.

Let $B(J,\mathbb{R})$ denote the space of bounded real-valued functions on J. Let $C(J,\mathbb{R})$, denote the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ and the multiplication " $\cdot$ " in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)|$$

and

$$(xy)(t) = x(t)y(t) \quad \text{for all} \quad t \in J.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with the above norm and the multiplication. By $L^1(J,\mathbb{R})$ we denote the set of Lebesgue integrable functions on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(t)|\,ds.$$

We need the following lemma in the sequel.

Lemma 1.7. *If $h \in L^1(J,\mathbb{R})$, then x is a solution of the IGDE*

$$\left.\begin{aligned} &\frac{d^2}{dt^2}\Big[\frac{x(t)}{f(t,x(t))}\Big] = \int_0^t h(s)\,ds \quad a.e.\ \ t \in J \\ &x(0) = x_0,\ x'(0) = x_1, \end{aligned}\right\} \tag{1.3}$$

if and only it is solution of the integral equation (IE for short)

$$x(t) = \big[f(t,x(t))\big]\Big(\frac{x_0}{f(0,x_0)} + ct + \int_0^t \frac{(t-s)^2}{2} h(s)\,ds\Big), \quad t \in J, \tag{1.4}$$

where $c = \dfrac{f(0,x_0)x_1 - x_0 f_1(0,x_0) - x_0 x_1 f_2(0,x_0)}{[f(0,x_0)]^2}$, and $\frac{\partial f}{\partial t} = f_1(t,x)$, $\frac{\partial f}{\partial x} = f_2(t,x)$.

We seek the solutions of the IE (1.4) in the space $C(J,\mathbb{R})$. We need the following definition in the sequel.

Definition 1.8. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be **Carathéodory** if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) $x \mapsto \beta(t,x)$ is continuous almost everywhere for $t \in J$.

Again a Carathódory function $\beta(t,x)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h_r(t), \quad a.e.\ t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally a Carathéodory function $\beta(t,x)$ is called L^1_X-Carathéodory if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that

$$|\beta(t,x)| \le h(t), \quad a.e.\ t \in J$$

for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a **bound function** of β.

We will need the following hypotheses in the sequel.

(H_1) The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $k \in B(J,\mathbb{R})$ such that $k(t) > 0$, a.e. $t \in J$ and

$$|f(t,x) - f(t,y)| \le k(t)\,|x-y|\,,\ \text{a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

(H_2) The function g is L^1_X-Carathéodory with bound function h.

(H_3) There exists a continuous and nondecreasing function $\Omega : [0,\infty) \to (0,\infty)$ and a function $\gamma \in L^1(J,\mathbb{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ satisfying

$$|g(t,x)| \le \gamma(t)\Omega\big(|x|\big), \quad \text{a.e. } t \in J,$$

for all $x \in \mathbb{R}$.

Theorem 1.9. *Assume that the hypotheses (H_1)-(H_3) hold. Suppose that*

$$\int_{K_1}^{\infty} \frac{ds}{\Omega(s)} > C_2\|\gamma\|_{L^1}, \tag{1.5}$$

where

$$K_1 = \frac{F\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T\right)}{1 - \|k\|\left[\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2\|h\|_{L^1}\right]},$$

$$K_2 = \frac{FT^2}{1 - \|k\|\left[\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2\|h\|_{L^1}\right]},$$

$\|k\|\left[\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2\|h\|_{L^1}\right] < 1$, $F = \max_{t\in J}|f(t,0)|$, *and* $\|k\| = \max_{t\in J}|k(t)|$. *Then the IGDE* (1.1) *has a solution on J.*

Proof. Set $X = C(J, \mathbb{R})$. Define two mappings A and B on X by

$$Ax(t) = f(t, x(t)), \quad t \in J, \tag{1.6}$$

and

$$Bx(t) = \frac{x_0}{f(0, x_0)} + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s, x(s))\, ds, \quad t \in J. \tag{1.7}$$

Obviously A and B define the operators $A, B : X \to X$. Then the FIE (1.4) is equivalent to the operator equation

$$x(t) = Ax(t)\, Bx(t), \quad t \in J. \tag{1.8}$$

We shall show that the operators A and B satisfy all the hypotheses of Theorem 1.1.

We first show that A is a Lipschitz on X. Let $x, y \in X$. Then by (H_1),

$$\begin{aligned} |Ax(t) - Ay(t)| &\le |f(t, x(t)) - f(t, y(t))| \\ &\le k(t)\,|x(t) - y(t)| \\ &\le \|k\|\,\|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain

$$\|Ax - Ay\| \le \|k\|\|x - y\|$$

for all $x, y \in X$. So A is a Lipschitz on X with Lipschitz constant $\|k\|$. Next we show that B is completely continuous on X. Using the standard arguments as in Granas et al. [8], it is shown that B is a continuous operator on X. Let S be a bounded set in X. We shall show that $B(X)$ is a uniformly bounded and equicontinuous set in X. Since $g(t, x(t))$ is L^1_X-Carathéodory, we have

$$\begin{aligned} |Bx(t)| &\le \left|\frac{x_0}{f(0, x_0)}\right| + |c|t + \frac{1}{2}\int_0^t |t-s|^2 |g(s, x(s))|\, ds \\ &\le \left|\frac{x_0}{f(0, x_0)}\right| + |c|T + \frac{1}{2}T^2\int_0^t h(s)\, ds \\ &\le \left|\frac{x_0}{f(0, x_0)}\right| + |c|T + \frac{1}{2}T^2\,\|h\|_{L^1}. \end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \le M$ for all $x \in S$, where $M = \left|\frac{x_0}{f(0,x_0)}\right| + |c|T + T^2\|h\|_{L^1}$. This shows that $B(X)$ is a uniformly bounded set in X. Now we show that $B(X)$ is an equicontinuous set. Let $t, \tau \in J$. Then for any $x \in X$ we have

$$\begin{aligned} |Bx(t) - Bx(\tau)| &\le |c||t-\tau| + \frac{1}{2}\left|\int_0^t (t-s)^2 g(s, x(s))\, ds - \int_0^\tau (\tau-s)^2 g(s, x(s))\, ds\right| \\ &\le |c||t-\tau| + \frac{1}{2}\left|\int_0^t (t-s)^2 g(s, x(s))\, ds - \int_0^t (\tau-s)^2 g(s, x(s))\, ds\right| \\ &\quad + \frac{1}{2}\left|\int_0^t (\tau-s)^2 g(s, x(s))\, ds - \int_0^\tau (\tau-s)^2 g(s, x(s))\, ds\right| \end{aligned}$$

$$
\begin{aligned}
&\leq |c||t-\tau| + \frac{1}{2}\left|\int_0^t [(t-s)^2-(\tau-s)^2] g(s,x(s))\,ds\right| \\
&\quad + \frac{1}{2}\left|\int_\tau^t (\tau-s)^2 g(s,x(s))\,ds\right| \\
&\leq |c||t-\tau| + \frac{1}{2}\int_0^T |(t-s)^2-(\tau-s)^2||g(s,x(s))|\,ds \\
&\quad + \frac{1}{2}T^2\left|\int_\tau^t |g(s,x(s))|\,ds\right| \\
&\leq |c||t-\tau| + T\int_0^T |t-\tau|h(s)\,ds + \frac{1}{2}T^2\left|\int_\tau^t h(s)\,ds\right| \\
&\leq (|c| + T\|h\|_{L^1})\,|t-\tau| + |p(t)-p(\tau)|
\end{aligned}
$$

where $p(t) = \frac{1}{2}T^2\int_0^t h(s)\,ds$. Therefore,

$$|Bx(t) - Bx(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

Hence $B(X)$ is an equicontinuous set and consequently $B(X)$ is relatively compact by Arzelà-Ascoli theorem. As a result B is a compact and continuous operator on X. Moreover $\alpha M < 1$ by assumption. Thus all the conditions of Theorem 1.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $x \in X$ be any solution to IGDE (1.1). Then we have, for any $\lambda \in (0,1)$,

$$x(t) = \lambda\left[f(t,x(t))\right]\left(\left|\frac{x_0}{f(0,x_0)}\right| + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s,x(s))\,ds\right)$$

for $t \in J$. Therefore,

$$
\begin{aligned}
|x(t)| \;&\leq\; \lambda\,|f(s,x(t)|\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|t + \frac{1}{2}\left|\int_0^t (t-s)^2 g(s,x(s))\,ds\right|\right) \\
&\leq\; \lambda\Big(|f(s,x(t)) - f(t,0)| + |f(t,0)|\Big) \\
&\qquad \times\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}\int_0^t |t-s|^2|g(s,x(s))|\,ds\right) \\
&\leq\; [k(t)\,|x(t)| + F]\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}\int_0^t |t-s|^2|g(s,x(s))|\,ds\right) \\
&\leq\; k(t)\,|x(t)|\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}\int_0^t |t-s|^2|g(s,x(s))|\,ds\right) \\
&\qquad + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}\int_0^t |t-s|^2|g(s,x(s))|\,ds\right) \\
&\leq\; \|k\|\,|x(t)|\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2\|h\|_{L^1}\right) \\
&\qquad + F\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2\int_0^t \gamma(s)\Omega(|x(t)|)\,ds\right). \qquad (1.9)
\end{aligned}
$$

Put $u(t)=\sup_{s\in[0,t]}|x(s)|$, for $t\in J$. Then we have $|x(t)|\le u(t)$ for all $t\in J$, and so, there is a point $t^*\in[0,t]$ such that $u(t)=|x(t^*)|$. From (1.9) it follows that

$$\begin{aligned}
u(t) &= |x(t^*)| \\
&\le \|k\|\,|x(t^*)|\left(\left|\frac{x_0}{f(0,x_0)}\right|+|c|T+\frac{1}{2}T^2\|h\|_{L^1}\right) \\
&\quad +F\left(\left|\frac{x_0}{f(0,x_0)}\right|+|c|T+\frac{1}{2}T^2\int_0^{t^*}\gamma(s)\Omega(|x(t)|)\,ds\right) \\
&\le \|k\|\,u(t)\left(\left|\frac{x_0}{f(0,x_0)}\right|+|c|T+\frac{1}{2}T^2\|h\|_{L^1}\right) \\
&\quad +F\left(\left|\frac{x_0}{f(0,x_0)}\right|+|c|T+\frac{1}{2}T^2\int_0^{t}\gamma(s)\Omega(u(s))\,ds\right).
\end{aligned}$$

Thus

$$u(t)=K_1+K_2\int_0^t\gamma(s)\Omega(u(s))\,ds,\quad t\in J.$$

Let

$$w(t)=K_1+K_2\int_0^t\gamma(s)\Omega(u(s))\,ds,\quad t\in J.$$

Then $u(t)\le w(t)$ and a direct differentiation of $w(t)$ yields

$$\left.\begin{aligned} w'(t) &\le K_2\gamma(t)\Omega(w(t)),\quad t\in J,\\ w(0) &= K_1,\end{aligned}\right\}\tag{1.10}$$

that is

$$\int_0^t\frac{w'(s)}{\Omega(w(s))}\,ds\le K_2\int_0^t\gamma(s)\,ds\le K_2\|\gamma\|_{L^1}.$$

A change of variables in the above integral gives that

$$\int_{K_1}^{w(t)}\frac{ds}{\Omega(s)}\le K_2\|\gamma\|_{L^1}<\int_{K_1}^{\infty}\frac{ds}{\Omega(s)}.$$

Now an application of mean value theorem yields that there is a constant $M>0$ such that $w(t)\le M$ for all $t\in J$. This further implies that

$$|x(t)|\le u(t)\le w(t)\le M,$$

for all $t\in J$. Thus the conclusion (ii) of Theorem 1.1 does not hold. Therefore the operator equation $AxBx=x$ and consequently the IGDE (1.1) has a solution on J. This completes the proof. □

2. EXISTENCE OF EXTREMAL SOLUTIONS

We equip the space $C(J,\mathbb{R})$ with the order relation $\le$ with the help of the cone defined by

$$K=\{x\in C(J,\mathbb{R}):x(t)\ge 0,\forall t\in J\}.\tag{2.1}$$

It is well known that the cone K is positive and normal in $C(J,\mathbb{R})$. We need the following definitions in the sequel.

Definition 2.1. A function $u \in AC^1(J,\mathbb{R})$ is called a lower solution of the IGDE (1.1) on J if

$$\frac{d^2}{dt^2}\left[\frac{u(t)}{f(t,u(t))}\right] \le \int_0^t g(s,u(s))ds, \ \text{ a.e. } \ t \in J, \ \text{and } u(0) \le x_0,\, u'(0) \le x_1.$$

Again a function $v \in AC^1(J,\mathbb{R})$ is called an upper solution of the IGDE (1.1) on J if

$$\frac{d^2}{dt^2}\left[\frac{v(t)}{f(t,v(t))}\right] \ge \int_0^t g(s,v(s))ds, \ \text{ a.e. } \ t \in J, \ \text{and } v(0) \ge x_0,\, v'(0) \ge x_1.$$

Definition 2.2. A solution x_M of the IGDE (1.1) is said to be maximal if for any other solution x to IGDE(1.1) one has $x(t) \le x_M(t)$, for all $t \in J$. Again a solution x_m of the IGDE (1.1) is said to be minimal if $x_m(t) \le x(t)$, for all $t \in J$, where x is any solution of the IGDE (1.1) on J.

2.1. **Carathéodory case.** We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g : J \times \mathbb{R} \to \mathbb{R}^+$, and

$$\frac{x_0}{f(0,x_0)} + \frac{f(0,x_0)x_1 - x_0 f_1(0,x_0) - x_0 x_1 f_2(0,x_0)}{[f(0,x_0)]^2} t \ge 0 \quad \text{for all} \quad t \in J.$$

(B_1) g is Carathéodory.

(B_2) The functions $f(t,x)$ and $g(t,x)$ are nondecreasing in x and y almost everywhere for $t \in J$.

(B_3) The IGDE (1.1) has a lower solution u and an upper solution v on J with $u \le v$.

(B_4) The function $\ell : J \to \mathbb{R}$ defined by

$$\ell(t) = |g(t,u(t))| = g(t,v(t)), \ t \in J$$

is Lebesgue integrable.

Remark 2.3. Assume that (B_2)-(B_4) hold. Then the function $t \mapsto g(t,x(t))$ is Lebesgue integrable on J and

$$|g(t,x(t))| \le \ell(t), \quad \text{a.e. } t \in J,$$

for all $x \in [u,v]$.

Theorem 2.4. *Suppose that the assumptions (H_1)-(H_3) and (B_0)-(B_4) hold. Further if*

$$\|k\| \left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2 \|\ell\|_{L^1}\right) < 1,$$

and ℓ is given in Remark 2.3, *then IGDE* (1.1) *has a minimal and a maximal positive solution on J.*

Proof. Now IGDE (1.1) is equivalent to FIE (1.4) on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (1.6) and (1.7) respectively. Then FIE (1.4) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_1) implies $A, B : [u,v] \to K$. Since the cone K in X is normal, $[u,v]$ is a norm bounded set in X. Now it is shown, as in the proof of Theorem 1.9, that A is a Lipschitz with a Lipschitz constant $\|\alpha\|$ and B is completely continuous operator on $[u,v]$. Again the hypothesis (B_2) implies that A and B are nondecreasing on $[u,v]$. To see this, let $x, y \in [u,v]$ be such that $x \le y$. Then by (B_2),

$$Ax(t) = f(t,x(t)) \le f(t,y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned} Bx(t) &= \frac{x_0}{f(0,x_0)} + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s,x(s))\,ds \\ &\le \frac{x_0}{f(0,x_0)} + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s,x(s))ds \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[u,v]$. Again Lemma 1.7 and hypothesis (B_3) together imply that

$$\begin{aligned} u(t) &\le [f(t,u(t))]\left(\frac{x_0}{f(0,x_0)} + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s,u(s))\,ds\right) \\ &\le [f(t,x(t))]\left(\frac{x_0}{f(0,x_0)} + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s,x(s))\,ds\right) \\ &\le [f(t,v(t))]\left(\frac{x_0}{f(0,x_0)} + ct + \frac{1}{2}\int_0^t (t-s)^2 g(s,v(s))\,ds\right) \\ &\le v(t), \end{aligned}$$

for all $t \in J$ and $x \in [u,v]$. As a result $u(t) \le Ax(t)Bx(t) \le v(t)$, for all $t \in J$ and $x \in [u,v]$. Hence $Ax\,Bx \in [u,v]$ for all $x \in [u,v]$.

Again

$$\begin{aligned} M &= \|B([u,v])\| \\ &= \sup\{\|Bx\| : x \in [u,v]\} \\ &\le \sup\left\{\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2 \sup_{t\in J}\int_0^t |g(s,x(s))|\,ds \;\Big|\; x \in [u,v]\right\} \\ &\le \left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2\int_0^T \ell(s)\,ds \\ &= \left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2\|\ell\|_{L^1}. \end{aligned}$$

Since
$$\alpha M \leq \|k\| \left(\left| \frac{x_0}{f(0,x_0)} \right| + |c|T + (1/2)T^2 \|\ell\|_{L^1} \right) < 1,$$
we apply Theorem 1.3 to the operator equation $Ax\,Bx = x$ to yield that the IGDE (1.1) has a minimal and a maximal positive solution on J. This completes the proof. □

2.2. **Discontinuous case.** We need the following definition in the sequel.

Definition 2.5. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be **Chandrabhan** if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) $x \mapsto \beta(t,x)$ is nondecreasing almost everywhere for $t \in J$.

Again a Chandrabhan function $\beta(t,x)$ is called L^1-Chandrabhan if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that
$$|\beta(t,x)| \leq h_r(t), \quad a.e.\ t \in J$$
for all $x \in \mathbb{R}$ with $|x| \leq r$.

Finally a Chandrabhan function $\beta(t,x)$ is called L^1_X-Chandrabhan if

(iv) there exists a function $h \in L^1(J,\mathbb{R})$ such that
$$|\beta(t,x)| \leq h(t), \quad a.e.\ t \in I$$
for all $x \in \mathbb{R}$.

For convenience, the function h is referred to as a **bound function** of β.

We consider the following hypotheses in the sequel.

(C_1) The function f is continuous on $J \times \mathbb{R}$.
(C_2) There is a function $k \in B(J,\mathbb{R})$ such that $k(t) > 0$, a.e. $t \in I$ and
$$|f(t,x) - f(t,y)| \leq k(t)\,|x-y|\,, \text{ a.e. } t \in J$$
for all $x, y \in \mathbb{R}$.
(C_3) The function $f(t,x)$ is nondecreasing in x almost everywhere for $t \in J$.
(C_4) The function g is L^1_X-Chandrabhan.

Theorem 2.6. *Suppose that the assumptions (B_0), (B_3), (B_4) and (C_1)-(C_4) hold. Then IGDE* (1.1) *has a minimal and a maximal positive solution on J.*

Proof. Now IGDE (1.1) is equivalent to FIE (1.4) on J. Let $X = C(J,\mathbb{R})$. Define two operators A and B on X by (1.6) and (1.7) respectively. Then FIE (1.4) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [u,v] \to K$. Since the cone K in X is normal, $[u,v]$ is a norm bounded set in X.

Step I : Next we show that A is completely continuous on $[a, b]$. Now the cone K in X is normal, so the order interval $[a, b]$ is norm-bounded. Hence there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. As f is continuous on compact $J \times [-r, r]$, it attains its maximum, say M. Therefore for any subset S of $[a, b]$ we have

$$\begin{aligned}
\|A(S)\| &= \sup\{\|Ax\| : x \in S\} \\
&= \sup\Big\{\sup_{t\in J} |f(t, x(t))| : x \in S\Big\} \\
&\leq \sup\Big\{\sup_{t\in J} |f(t, x)| : x \in [-r, r]\Big\} \\
&\leq M.
\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the function $f(t, x)$ is uniformly continuous on $[0, T] \times [-r, r]$. Therefore for any $t, \tau \in [0, T]$ we have

$$|f(t, x) - f(\tau, x)| \to 0 \text{ as } t \to \tau$$

for all $x \in [-r, r]$. Similarly for any $x, y \in [-r, r]$

$$|f(t, x) - f(t, y)| \to 0 \text{ as } x \to y$$

for all $t \in [0, T]$. Hence for any $t, \tau \in [0, T]$ and for any $x \in S$ one has

$$\begin{aligned}
|Ax(t) - Ax(\tau)| &= |f(t, x(t)) - f(\tau, x(\tau))| \\
&\leq |f(t, x(t)) - f(\tau, x(t)| + |f(\tau, x(t)) - f(\tau, x(\tau))| \\
&\to 0 \quad \text{as} \quad t \to \tau.
\end{aligned}$$

This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[a, b]$.

Step II : Next we show that B is totally bounded operator on $[a, b]$. To finish, we shall show that $B(S)$ is uniformly bounded and equi-continuous set in X for any subset S of $[a, b]$. Since the cone K in X is normal, the order interval $[a, b]$ is norm-bounded. Let $y \in B(S)$ be arbitrary. Then,

$$y(t) = \frac{x_0}{f(0, x_0)} + ct + \frac{1}{2}\int_0^t (t - s)^2 g(s, x(s))\, ds$$

for some $x \in S$. By hypothesis (C_4) one has

$$\begin{aligned}
|y(t)| &\leq \left|\frac{x_0}{f(0, x_0)}\right| + |c|T + \frac{1}{2}T^2 \int_0^t |g(s, x(s))|\, ds \\
&\leq \left|\frac{x_0}{f(0, x_0)}\right| + |c|T + \frac{1}{2}T^2 \int_0^t \ell(s)\, ds
\end{aligned}$$

$$\leq \left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2\|\ell\|_{L^1}.$$

Taking the supremum over t,

$$\|y\| \leq \left|\frac{x_0}{f(0,x_0)}\right| + |c|T + \frac{1}{2}T^2\|\ell\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Similarly let $t, \tau \in J$. Then for any $y \in B(S)$,

$$\begin{aligned}
|y(t) - y(\tau)| &\leq |c|\,|t-\tau| + \frac{1}{2}\left|\int_0^t (t-s)^2 g(s,x(s))\,ds - \int_0^\tau (\tau-s)^2 g(s,x(s))\,ds\right| \\
&\leq |c|\,|t-\tau| + \frac{1}{2}\left|\int_0^t (t-s)^2 g(s,x(s))\,ds - \int_0^t (\tau-s)^2 g(s,x(s))\,ds\right| \\
&\quad + \frac{1}{2}\left|\int_0^t (\tau-s)^2 g(s,x(s))\,ds - \int_0^\tau (\tau-s)^2 g(s,x(s))\,ds\right| \\
&\leq |c|\,|t-\tau| + \frac{1}{2}\left|\int_0^t |(t-s)^2 - (\tau-s)^2| g(s,x(s))\,ds\right| \\
&\quad + \frac{1}{2}\left|\int_\tau^t (\tau-s)^2 g(s,x(s))\,ds\right| \\
&\leq |c|\,|t-\tau| + \frac{1}{2}\int_0^T |(t-s)^2 - (\tau-s)^2||g(s,x(s))|\,ds \\
&\quad + \frac{1}{2}T^2\left|\int_\tau^t |g(s,x(s))|\,ds\right| \\
&\leq |c|\,|t-\tau| + T\int_0^T |t-\tau|\ell(s)\,ds + \frac{1}{2}T^2\left|\int_\tau^t \ell(s)\,ds\right| \\
&\leq (|c| + T\|\ell\|_{L^1})\,|t-\tau| + |p(t) - p(\tau)|
\end{aligned}$$

where $p(t) = \dfrac{T^2}{2}\displaystyle\int_0^t \ell(s)\,ds$. Since the function p is continuous on compact interval J, it is uniformly continuous, and therefore we have

$$|y(t) - y(\tau)| \to 0 \quad \text{as} \quad t \to \tau,$$

for all $y \in B(S)$. Hence $B(S)$ is an equi-continuous set in X. Thus B is totally bounded in view of Arzelà-Ascoli theorem.

Thus all the conditions of Theorem 1.4 are satisfied and hence an application of it yields that the IGDE (1.1) has a maximal and a minimal positive solution on J. □

Theorem 2.7. *Suppose that the assumptions* (B_0), (B_3) *and* (C_1)-(C_4) *hold. Further if*

$$\|k\|\left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2\|\ell\|_{L^1}\right) < 1,$$

and ℓ is given in Remark 2.3, *then IGDE* (1.1) *has a minimal and a maximal positive solution on J.*

Proof. Now IGDE (1.1) is equivalent to FIE (1.4) on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on X by (1.6) and (1.7) respectively. Then IE (1.4) is transformed into an operator equation $Ax(t)\,Bx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [u, v] \to K$. Since the cone K in X is normal, $[u, v]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 1.9 and Theorem 1.5 that the operator A is a Lipschitz with a Lipschitz constant $\alpha = \|k\|$ and B is totally bounded with $M = \|B([u, v])\| = \left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2\|\ell\|_{L^1}$. Since $\alpha M = \|k\| \left(\left|\frac{x_0}{f(0,x_0)}\right| + |c|T + (1/2)T^2\|\ell\|_{L^1}\right) < 1$, the desired conclusion follows by an application of Theorem 1.5. This completes the proof. □

3. AN EXAMPLE

Given the closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the nonlinear IGDE

$$\left.\begin{aligned} \frac{d^2}{dt^2}\left[\frac{x(t)}{f(t, x(t))}\right] &= \int_0^t \left(\frac{p(s)\,x(s)}{1 + |x(s)|}\right) ds, \quad \text{a.e. } t \in J \\ x(0) = 0,&\ x'(0) = 1, \end{aligned}\right\} \tag{3.1}$$

where $p \in L^1(J, \mathbb{R})$ and the function $f : J \times \mathbb{R} - \{0\} \to \mathbb{R}$ is defined by

$$f(t, x) = 1 + \alpha\, t\, |x|,$$

for $\alpha > 0$. Obviously $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$. Define a function $g : J \times \mathbb{R} \to \mathbb{R}$ by

$$g(t, x) = \frac{p(t)x}{1 + |x|}.$$

It is easy to verify that f is continuous and Lipschitz on $J \times \mathbb{R}$ with a Lipschitz constant α. Further $g(t, x)$ is L^1_X-Carathéodory with the bound function $h(t) = p(t)$ on J. Therefore if $\alpha(1 + \frac{1}{2}\|p\|_{L^1}) < 1$, then by Theorem 1.9, the IGDE (3.1) has a solution on J, because the function Ω satisfies condition (1.5) with $\gamma(t) = p(t)$ for all $t \in J$ and $\Omega(r) = r$ for all $r \in \mathbb{R}^+$.

REFERENCES

[1] B. C. Dhage, *On α-condensing mappings in Banach algebras*, The Math. Student **63**(1994), 146-152.

[2] B. C. Dhage, *Fixed point theorems in ordered Banach algebras and applications*, PanAmer. Math. J. **9**(4) (1999), 93–102.

[3] B. C. Dhage, *A nonlinear alternative in Banach algebras with applications to functional differential equations*, Nonlinear Funct. Anal. & Appl. **8** (40 (2004), 563-575.

[4] B. C. Dhage, *On a fixed point theorem in Banach algebras with applications*, Appl. Math. Lett. **18** (2005), 273-280.

[5] B. C. Dhage, *Some algebraic fixed point theorems for multi-valued mappings with applications*, Discuss. Math. Diff. Inclusions Control & Otimization bf 26, (2006) (in press)

[6] B. C. Dhage and B. D. Karande, *First order integro-differential equations in Banach algebras involving Carathéodory and discontinuous nonlinearities*, Electron. J. Qual. Theory Differ. Equ. 2005, No 21, 16 pp. (electronic).

[7] B. C. Dhage and D. O'Regan, *A fixed point theorem in Banach algebras with applications to nonlinear integral equation*, Functional Diff. Equations **7**(3-4)(2000), 259-267.

[8] A. Granas, R. B. Guenther and J. W. Lee, *Some general existence principles for Carathéodory theory of nonlinear differential equations*, J. Math. Pures et Appl. **70** (1991), 153-196.

[9] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Spaces*, Academic Press, New York, 1988.