RIGIDITY OF HOLOMORPHIC GENERATORS AND ONE-PARAMETER SEMIGROUPS

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ABSTRACT. In this paper we establish a rigidity property of holomorphic generators by using their local behavior at a boundary point \( \tau \) of the open unit disk \( \Delta \). Namely, if \( f \in \text{Hol}(\Delta, \mathbb{C}) \) is the generator of a one-parameter continuous semigroup \( \{F_t\}_{t \geq 0} \), we show that the equality \( f(z) = o(|z-\tau|^3) \) when \( z \to \tau \) in each non-tangential approach region at \( \tau \) implies that \( f \) vanishes identically on \( \Delta \). Note, that if \( F \) is a self-mapping of \( \Delta \) then \( f = I - F \) is a generator, so our result extends the boundary version of the Schwarz Lemma obtained by D. Burns and S. Krantz. We also prove that two semigroups \( \{F_t\}_{t \geq 0} \) and \( \{G_t\}_{t \geq 0} \), with generators \( f \) and \( g \) respectively, commute if and only if the equality \( f = \alpha g \) holds for some complex constant \( \alpha \). This fact gives simple conditions on the generators of two commuting semigroups at their common null point \( \tau \) under which the semigroups coincide identically on \( \Delta \).

Keywords. holomorphic generators, boundary Schwarz Lemma, commuting family

AMS (MOS) Subject Classification. 30D05, 32H12, 47B33, 47H20

1. INTRODUCTION

Let \( \Delta = \{z \in \mathbb{C} : |z| < 1\} \) be the open unit disk in the complex plane \( \mathbb{C} \), and let \( H = \{z \in \mathbb{C} : \Re z > 0\} \) be the right half-plane. We denote by \( \text{Hol}(\Delta, D) \) the set of all holomorphic functions on \( \Delta \) which map \( \Delta \) into a set \( D \subset \mathbb{C} \), and by \( \text{Hol}(\Delta) \) the set of all holomorphic self-mappings of \( \Delta \), i.e., \( \text{Hol}(\Delta) = \text{Hol}(\Delta, \Delta) \).

The problem of finding conditions for a holomorphic function \( F \) to coincide identically with a given holomorphic function \( G \) when they have a similar behavior on some subset of \( \overline{\Delta} \), has been studied by many mathematicians.

The following assertions are classical:
If $F$ and $G$ are holomorphic in $\Delta$ and $F = G$ on a subset of $\Delta$ that has a nonisolated point, then $F \equiv G$ on $\Delta$ (Vitali’s uniqueness principle).

If $F$ and $G$ are holomorphic in $\Delta$ and continuous on $\overline{\Delta}$, and $F = G$ on some arc $\gamma$ of the boundary $\partial \Delta$, then $F \equiv G$ on $\Delta$.

From the point of view of complex dynamics it is natural to study conditions on derivatives of $F$ and $G$ at specific points to conclude that $F \equiv G$.

If, for example, $G$ is the identity mapping $I$ and $2$ is the Denjoy–Wolff point of $F \in \text{Hol}(\Delta)$, then the equalities $F(\tau) = G(\tau)$ and $F'(\tau) = G'(\tau)$ imply $F \equiv G$ by the Schwarz Lemma. The same conclusion holds for an arbitrary holomorphic function $G$ on $\Delta$, if $F$ commutes with $G$ and satisfies the conditions $F(\tau) = G(\tau) = \tau$ and $F'(\tau) = G'(\tau) \neq 0$ (see, for instance, [10], [7]).

Different “identity principles” have recently been studied by several mathematicians under suitable boundary conditions. In general, the following three cases are considered.

(A) $G$ is the identity mapping;

(B) $G$ is an arbitrary self-mapping of $\Delta$, and $F$ commutes with $G$, i.e., $F \circ G = G \circ F$;

(C) $G$ is a constant mapping.

Regarding Case A the following result is due to D. Burns and S. Krantz.

**Theorem A** ([8]). Let $F \in \text{Hol}(\Delta)$ and

$$F(z) = 1 + (z - 1) + O((z - 1)^4) .$$

Then $F \equiv I$.

For Case B a uniqueness theorem was given by R. Tauraso in [19] (see also [7]). To formulate this result we need the following notation. Let $F \in \text{Hol}(\Delta)$ and $\tau \in \partial \Delta$. We say that $F \in C_K^m(\tau)$ if it admits the following representation

$$F(z) = \tau + F'(\tau)(z - \tau) + \cdots + \frac{F^{(m)}(\tau)}{m!}(z - \tau)^m + o(|z - \tau|^m)$$

when $z \to \tau$ in each non-tangential approach region at $\tau$, $z \in D_\alpha(\tau) := \{ z : |z - \tau| < \frac{\alpha}{2}(1 - |z|^2) \}$, $\alpha > 1$, (sometimes this domain is referred to as the Koranyi domain). Moreover, we say that $F \in C_K^m(\tau)$ if the limit is taken in the full disk.

**Theorem B** ([19]). Let $F, G \in \text{Hol}(\Delta)$ be commuting functions with a common Denjoy–Wolff point $\tau \in \partial \Delta$. If one of the following conditions holds then $F \equiv G$.

(i) $F'(\tau) = G'(\tau) < 1$;
(ii) $F \in C^2(\tau)$, $G \in C_K^2(\tau)$, $F''(\tau) = 1$, $F''(\tau) = G''(\tau) \neq 0$ and $\text{Re}\tau F''(\tau) > 0$;
(iii) $F, G \in C^2(\tau)$, $F'(\tau) = 1$, $F''(\tau) = G''(\tau) \neq 0$ and $\text{Re}\tau F''(\tau) = 0$;
(iv) $F \in C^3(\tau), \ G \in C^3_K(\tau), \ F''(\tau) = 1, \ F''(\tau) = G''(\tau) = 0$ and $F'''(\tau) = G'''(\tau)$.

For Case C, when $G$ is a constant mapping, the following fact is an immediate consequence of the Julia–Wolff–Carathéodory Theorem.

- If $F \in \text{Hol}(\Delta, \overline{\Delta}),$ then the conditions $\lim_{r \to 1^-} F(r\tau) = \tau$ and $\lim_{r \to 1^-} F'(r\tau) = 0$ for some $\tau \in \partial \Delta$ imply that $F \equiv \tau$.

In fact, the consideration of holomorphic functions $f$ which are not necessarily self-mappings is more relevant in this situation. Various results in this direction were established by S. Migliorini and F. Vlacci in [14].

In what follows we denote by symbol $\angle \lim_{z \to \tau}$ the angular limit of a function defined in $\Delta$ at a boundary point $\tau \in \partial \Delta$.

**Theorem C** (see [14]). Let $\tau \in \partial \Delta$.

If $f \in \text{Hol}(\Delta, \overline{\Delta})$, then

$$\angle \lim_{z \to \tau} \frac{f(z)}{z - \tau} = 0$$

implies that $f \equiv 0$.

More general, if $f \in \text{Hol}(\Delta, \mathbb{C})$, and $f(\Delta)$ is contained in a wedge of angle $\pi \alpha$, $0 < \alpha \leq 2$, with vertex at the origin, then the condition

$$\angle \lim_{z \to \tau} \frac{f(z)}{(z - \tau)^\alpha} = 0$$

implies that $f \equiv 0$.

Although the classes $\text{Hol}(\Delta)$ of holomorphic self-mappings of $\Delta$ and $\text{Hol}(\Delta, H)$ of functions with positive real part are connected by the composition with the Cayley transform, Theorem A is not a direct consequence of Theorem C, and conversely.

In this note we find rigidity principles for some classes of holomorphic functions produced by continuous dynamical systems, which are related to both $\text{Hol}(\Delta)$ and $\text{Hol}(\Delta, H)$. In particular, by this way one can establish a bridge between Theorems A and C.

We consider, inter alia, the class of mappings $F \in \text{Hol}(\Delta, \mathbb{C})$ which are continuous on $\overline{\Delta}$ and satisfy the boundary flow-invariance condition

$$\Re F(z) \overline{z} \leq 1, \ z \in \partial \Delta.$$  

In particular, each function $F \in \text{Hol}(\Delta)$ which is continuous on $\overline{\Delta}$ belongs to this class.

Condition (4) can be rewritten in the form

$$\Re f(z) \overline{z} \geq 0, \ z \in \partial \Delta,$$
where

\[ f(z) = z - F(z). \]

Note that each mapping \( f \) satisfying (5) belongs to the class \( \mathcal{G}(\Delta) \) of so-called semigroup generators on \( \Delta \) (see Corollary 4.5 in [15] or Condition (3), p. 41 in [4]).

Our main purpose is to establish boundary conditions for a function \( f \in \mathcal{G}(\Delta) \) to vanish on \( \Delta \) identically.

First, we recall that a family \( S = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Delta) \) is said to be a one-parameter continuous semigroup on \( \Delta \) if

(i) \( F_t(F_s(z)) = F_{t+s}(z) \) for all \( t, s \geq 0 \),

(ii) \( \lim_{t \to 0^+} F_t(z) = z \) for all \( z \in \Delta \).

Furthermore, it follows from a result of E. Berkson and H. Porta [6] that each continuous semigroup is differentiable in \( t \in \mathbb{R}^+ = [0, \infty) \), (see also [1] and [16]). So, for each continuous semigroup \( S = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Delta) \), the limit

\[ \lim_{t \to 0^+} \frac{z - F_t(z)}{t} = f(z), \quad z \in \Delta, \]

exists and defines a holomorphic mapping \( f \in \text{Hol}(\Delta, \mathbb{C}) \). This mapping \( f \) is called the \textbf{(infinitesimal) generator of} \( S = \{F_t\}_{t \geq 0} \). Moreover, the function \( u(= u(t, z)) \), \( (t, z) \in \mathbb{R}^+ \times \Delta \), defined by \( u(t, z) = F_t(z) \) is the unique solution of the Cauchy problem

\[ \begin{cases} 
\frac{\partial u(t, z)}{\partial t} + f(u(t, z)) = 0, \\
u(0, z) = z, \quad z \in \Delta.
\end{cases} \]

The class of all holomorphic generators on \( \Delta \) is denoted by \( \mathcal{G}(\Delta) \).

Note, that if \( F \in \text{Hol}(\Delta) \), then the function \( f = I - F \) belongs to \( \mathcal{G}(\Delta) \) (see Proposition 4.3 in [15] and Corollary 3.3.1 in [17]).

The following assertion combines characterizations of the class \( \mathcal{G}(\Delta) \) obtained in [2], [3] and [6].

\textbf{Proposition 1.} Let \( f \in \text{Hol}(\Delta, \mathbb{C}) \). The following are equivalent:

(i) \( f \) is a semigroup generator on \( \Delta \);

(ii) \( \text{Re} f(z) \overline{\tau} \geq \text{Re} f(0) \overline{\tau} (1 - |z|^2) \) for all \( z \in \Delta \);

(iii) there exists a unique point \( \tau \in \overline{\Delta} \) such that

\[ f(z) = (z - \tau)(1 - \overline{\tau}z)g(z), \quad z \in \Delta, \]

where \( g \in \text{Hol}(\Delta, \mathbb{C}), \text{Re} g(z) \geq 0 \).

(iv) \( f \) admits the representation

\[ f(z) = a - \overline{a}z^2 + pz(z), \]
where \( a \in \mathbb{C} \) and \( p \in \text{Hol}(\Delta, \mathbb{C}) \) with \( \text{Re} \, p(z) \geq 0 \).

**Remark 1.** The point \( \tau \) in (9) is the Denjoy–Wolff point of the semigroup \( \{F_t\}_{t \geq 0} \) generated by \( f \). If \( \tau \in \Delta \) then \( f(0) = 0 \) and \( \text{Re} \, f'(\tau) \geq 0 \). If \( \tau \in \partial \Delta \) then the angular limit \( \angle \lim_{z \to \tau} \frac{f(z)}{z - \tau} =: f'(\tau) \) exists and is a nonnegative real number (see [11]).

### 2. RIGIDITY OF INFINITESIMAL GENERATORS

**Theorem 1.** Let \( f \in \mathcal{G}(\Delta) \). Suppose that for some \( \tau \in \partial \Delta \)
\[
f(z) = a(z - \tau)^3 + o(|z - \tau|^3)
\]
when \( z \to \tau \) in each non-tangential approach region at \( \tau \). Then \( a\tau^2 \) is a nonnegative real number. Moreover, \( a = 0 \) if and only if \( f \equiv 0 \).

To prove Theorem 1 we need the following lemma.

**Lemma 1.** Let \( g \in \text{Hol}(\Delta, \overline{H}) \). Then for each \( \tau \in \partial \Delta \) the limit
\[
k = \angle \lim_{z \to \tau} \frac{g(z)}{1 - \overline{\tau}z}
\]
is either a nonnegative real number or infinity. Moreover, \( g \equiv 0 \) if and only if \( k = 0 \).

**Proof.** Denote by \( C_\tau(z) = \frac{z - \tau}{\tau + z} \) the Cayley transform and set \( h = C_\tau^{-1} \circ g \in \text{Hol}(\Delta, \overline{\Delta}) \).

By the Julia–Wolff–Carathéodory theorem the limit
\[
\beta_h = \angle \lim_{z \to \tau} \frac{\tau - h(z)}{\tau - z}
\]
exists and is either a nonnegative real number or infinity. Moreover, \( \beta_h = 0 \) if and only if \( h \equiv \tau \).

For any \( z \in \Delta \) we have
\[
g(z) = \frac{\tau - h(z)}{\tau - z} \cdot \frac{\tau}{\tau + h(z)}.
\]
Hence, \( k = 0 \) if and only if \( \beta_h = 0 \), and therefore \( g \equiv 0 \).

If \( \beta_h \) is a positive real number, \( \beta_h > 0 \), then \( \angle \lim_{z \to \tau} h(z) = \tau \) and, consequently,
\[
k = \angle \lim_{z \to \tau} \frac{\tau - h(z)}{\tau - z} \cdot \angle \lim_{z \to \tau} \frac{\tau}{\tau + h(z)} = \frac{\beta_h}{2} > 0.
\]

Let \( \beta_h = \infty \). Since \( \text{Re} \, \frac{\tau}{\tau + h(z)} \geq \frac{1}{2} \), formula (11) implies that \( k = \infty \). \( \Box \)

**Alternative proof.** If \( g \neq 0 \), then the function \( p \) defined by \( p(z) := \frac{1}{g(z)} \) belongs to \( \text{Hol}(\Delta, H) \). It is easy to see that for all \( \zeta \in \partial \Delta \) the expression \( \frac{(1 - \zeta)(1 + \overline{\zeta})}{1 - \overline{z}\zeta} \) is bounded on each non-tangential approach region at \( \tau \). Then it follows by the Riesz–Herglotz formula that
\[
\angle \lim_{z \to \tau} (1 - z\overline{\tau})p(z) = \angle \lim_{z \to \tau} \int_{\partial \Delta} \frac{(1 - z\overline{\tau})(1 + \overline{\zeta})}{1 - z\zeta} \, dm_p(\zeta) = 2m_p(\tau) \geq 0,
\]
where \( dm_p \) is a probability measure on \( \partial \Delta \). Setting \( k = \frac{1}{2m_p(\tau)} \) we get our assertion. □

**Proof of Theorem 1.** Since
\[
\angle \lim_{z \to \tau} \frac{f(z)}{z - \tau} = 0,
\]
it follows from [11] that \( \tau \in \partial \Delta \) is the Denjoy–Wolff point for the semigroup \( \{F_t\}_{t \geq 0} \) generated by \( f \). Then by Proposition 1 the function \( f \) admits the representation (9):
\[
f(z) = (z - \tau)(1 - \overline{\tau}g(z)
\]
with some \( g \in \text{Hol}(\Delta, \overline{\mathbb{H}}) \). Hence, by Lemma 1
\[
a\tau^2 = \tau^2 \angle \lim_{z \to \tau} \frac{f(z)}{(z - \tau)^3} = \angle \lim_{z \to \tau} \frac{g(z)}{1 - \overline{\tau}z} = k \geq 0.
\]
Obviously, \( a = 0 \) if and only if \( k = 0 \). In this case \( g \equiv 0 \), so \( f \equiv 0 \). □

**Corollary 1** (cf. Theorem 5 in [7]). Let \( F \in \text{Hol}(\Delta, \mathbb{C}) \) be continuous on \( \overline{\Delta} \) and satisfy the boundary condition
\[
\text{Re } F(z) \overline{z} \leq 1, \ z \in \partial \Delta.
\]
If \( F \) admits the representation
\[
F(z) = \tau + (z - \tau) + b(z - \tau)^3 + o \left( |z - \tau|^3 \right)
\]
when \( z \to \tau \) in each non-tangential approach region at some point \( \tau \in \partial \Delta \), then
\[
br \tau^2 \leq 0.
\]
Moreover, \( b = 0 \) if and only if \( F \equiv I \).

As a consequence of Lemma 1 we also obtain the following assertion.

**Corollary 2.** Let \( f \in G(\Delta) \) be such that \( f(\tau) = 0 \) for some \( \tau \in \partial \Delta \) and \( f(0) = a \in \mathbb{C} \). Suppose that \( f \) has a finite angular derivative at \( \tau \). Then \( f'(\tau) \) is a real number with \( f'(\tau) \leq -2\text{Re}(\overline{\tau}a) \). Moreover, \( f'(\tau) = -2\text{Re}(\overline{\tau}a) \) if and only if \( f \) generates a group of automorphisms.

**Proof.** By Proposition 1 (iv) \( f \) admits the representation
\[
f(z) = a - \overline{\tau}z^2 +zp(z), \ z \in \Delta,
\]
where \( p \in \text{Hol}(\Delta, \mathbb{C}) \) with \( \text{Re } p(z) \geq 0 \).

Since \( f(\tau) = 0 \), we have \( p(\tau) = \overline{a\tau} - a\overline{\tau} = 2i \text{Im}(\overline{a\tau}) \) is pure imaginary.

Then it follows from (12), that
\[
f'(\tau) = \angle \lim_{z \to \tau} \frac{a - \overline{\tau}z^2 +zp(z)}{z - \tau} = -2\text{Re}(a\overline{\tau}) + \angle \lim_{z \to \tau} \frac{p(z) - 2i \text{Im}(\overline{a\tau})}{z\overline{\tau} - 1}.
\]
Applying Lemma 1 to the function \( g(z) = p(z) - 2i \text{Im}(\overline{a\tau}) \), we get \( f'(\tau) \leq -2\text{Re}(a\overline{\tau}) \).
Moreover, \( f'(\tau) = -2 \text{Re}(a\tau) \) if and only if \( p \equiv 2i \text{Im}(\bar{\tau} r) \), i.e., \( f(z) = a + 2i \text{Im}(\bar{a} \tau) \cdot z - \bar{a} z^2 \).

By Proposition 3.5.1 in [17] (see also Remark 2 in [4]) each function of the form \( f(z) = a + ibz - \bar{a} z^2 \), with \( a \in \mathbb{C} \) and \( b \in \mathbb{R} \), generates a group of automorphisms of \( \Delta \). The proof is complete.

**Corollary 3.** Let \( F \in \text{Hol}(\Delta) \) be such that \( F(\tau) = \tau \) and \( F(0) = a, a \in \Delta \). Suppose that \( F \) has a finite angular derivative at \( \tau \). Then \( F'(\tau) \geq 1 + 2 \text{Re}(\bar{\tau} r) \).

**Proof.** By a result in [15, Proposition 4.3] (see also [17, Corollary 3.3.1]) the function \( f(z) = z - F(z), z \in \Delta \) is a generator of a one-parameter semigroup. By our assumptions we have \( f(\tau) = 0 \) and \( f(0) = -a \). Hence, by Corollary 2 \( f'(\tau) \leq -2 \text{Re}(\bar{\tau} r) \), and \( F'(\tau) \geq 1 + 2 \text{Re}(\bar{\tau} r) \).

Now let us consider the class of functions \( f \in \text{Hol}(\Delta, \mathbb{C}) \) which are continuous on \( \overline{\Delta} \) and satisfy the boundary condition
\[
(13) \quad \text{Re} f(z) \bar{\tau} \geq |f(z)| \cos \frac{\alpha \pi}{2} \quad \text{for all} \quad z \in \partial \Delta,
\]
for some \( \alpha \in (0, 2] \). As we already mentioned if \( \alpha \leq 1 \) then condition (13) implies \( f \in \mathcal{G}(\Delta) \) (cf. Proposition 1 (ii)). Conversely, if \( f \in \mathcal{G}(\Delta) \) is continuous on \( \overline{\Delta} \), then (13) holds with \( \alpha = 1 \). So, this class generalizes in a sense the class of holomorphic generators which are continuous on \( \overline{\Delta} \).

**Theorem 2.** Let \( f \in \text{Hol}(\Delta, \mathbb{C}) \) be continuous on \( \overline{\Delta} \) and satisfy the condition (13). Then the condition
\[
(14) \quad \lim_{\begin{array}{c} z \to \tau \\ z \in \overline{\Delta} \end{array}} \frac{f(z)}{(z - \tau)^{2+\alpha}} = 0 \quad \text{for some} \quad \tau \in \partial \Delta
\]
implies that \( f \equiv 0 \).

**Proof.** Denote
\[
g(z) = \frac{f(z)}{(z - \tau)(1 - \bar{\tau} z)}.
\]
The continuity of \( f \) and (14) imply that this function is continuous (consequently, bounded) on \( \overline{\Delta} \).

Now we rewrite (13) in the form:
\[
- \text{Re} \left[ \bar{\tau} (\tau - z)^2 g(z) \bar{\tau} \right] \geq |\tau - z|^2 \cdot |g(z)| \cdot \cos \frac{\alpha \pi}{2}, \quad z \in \partial \Delta.
\]
Hence,
\[
\text{Re} g(z) \geq |g(z)| \cdot \cos \frac{\alpha \pi}{2}, \quad z \in \partial \Delta \setminus \{\tau\}.
\]
This inequality also holds at the point \( \tau \) because of the continuity of \( g \).
It follows from the subordination principle for subharmonic functions (see, for example, [12, p. 396]) that the latter inequality holds for all \( z \in \Omega \). Geometrically this fact means that \( g \) maps \( \Delta \) into the sector \( A_\alpha \), where
\[
A_\alpha = \left\{ w \in \mathbb{C} : |\arg w| < \frac{\alpha \pi}{2}, \alpha \in (0, 2] \right\}.
\]

Suppose that there exists \( z \in \Delta \) such that \( w = g(z) \in \partial A_\alpha \). Then by the maximum principle \( g \equiv \text{const} = w \) and \( f(z) = w\tau(z - \tau)^2 \). In this case \( w \) must be zero, since otherwise we get a contradiction with (14). Hence, either \( w = 0 \) or \( g(\Delta) \subset A_\alpha \).

If \( w = 0 \) then \( f \equiv 0 \) and we are done.

Let now \( g(\Delta) \subset A_\alpha \). Equality (14) implies that
\[
\angle \lim_{z \to \tau} \frac{g(z)}{(z - \tau)^\alpha} = -\tau \angle \lim_{z \to \tau} \frac{f(z)}{(z - \tau)^{2+\alpha}} = 0.
\]

Applying Theorem C we get \( g \equiv 0 \), hence \( f \equiv 0 \).

**Corollary 4.** Let \( F \in \text{Hol}(\Delta, \mathbb{C}) \) be continuous on \( \overline{\Delta} \) and satisfy the boundary condition
\[
\text{Re } F(z)z \leq 1 - |F(z) - z| \cos \frac{\alpha \pi}{2}, \quad z \in \partial \Delta,
\]
for some \( \alpha \in (0, 2] \). If there exists \( \tau \in \partial \Delta \) such that
\[
F(z) = \tau + (z - \tau) + o(|z - \tau|^{2+\alpha})
\]
when \( z \to \tau \), then \( F \equiv I \).

### 3. COMMUTING SEMIGROUPS

**Theorem 3.** Let \( f \) and \( g \) be generators of one-parameter commuting semigroups \( \{F_t\}_{t \geq 0} \) and \( \{G_t\}_{t \geq 0} \), respectively, and \( f(\tau) = 0 \) at some point \( \tau \in \overline{\Delta} \).

(i) Let \( \tau \in \Delta \). If \( f'(\tau) = g'(\tau) \) then \( f \equiv g \).

(ii) Let \( \tau \in \partial \Delta \). Suppose \( f \) and \( g \) admit the following representations:
\[
f(z) = f'(\tau)(z - \tau) + \ldots + \frac{f^{(m)}(\tau)}{m!}(z - \tau)^m + o(|z - \tau|^m)
\]
and
\[
g(z) = g(\tau) + g'(\tau)(z - \tau) + \ldots + \frac{g^{(m)}(\tau)}{m!}(z - \tau)^m + o(|z - \tau|^m)
\]
when \( z \to \tau \) along some curve lying in \( \Delta \) and ending at \( \tau \). If \( f^{(m)}(\tau) = g^{(m)}(\tau) \neq 0 \), then \( f \equiv g \).
Remark 2. If $\tau \in \partial \Delta$ is the Denjoy–Wolff point of a semigroup generated by a mapping $h \in G(\Delta)$, then $h$ admits the expansion

$$h(z) = h'(\tau)(z - \tau) + o(z - \tau)$$

when $z \to \tau$ in each non-tangential approach region at $\tau$ and $h'(\tau) = \angle \lim_{z \to \tau} h'(z)$. Moreover, in this case $h'(\tau)$ is a non-negative real number which is zero if and only if $h$ generates a semigroup of parabolic type (see [11]).

Therefore, if $f$ (or $g$) in Theorem 3 generates a semigroup of hyperbolic type with the Denjoy–Wolff point $\tau \in \partial \Delta$ then the condition $f'(\tau) = g'(\tau)$ is enough to provide that $f \equiv g$.

Remark 3. As a matter of fact, if $f$ and $g$ have expansion (16) and (17) when $z \to \tau$ in each non-tangential approach region at $\tau \in \partial \Delta$ up to the third order $m = 3$, such that $f'(\tau) = g'(\tau)$, $f''(\tau) = g''(\tau)$ and $f'''(\tau) = g'''(\tau)$ then $f \equiv g$.

If, in particular, $f^{(i)}(\tau) = g^{(i)}(\tau) = 0$, $i = 1, 2, 3$, then both $f$ and $g$ equal zero identically by Theorem 1.

Theorem 3 is a consequence of the following more general assertion.

Define two linear semigroups $\{A_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$ of composition operators on $\text{Hol}(\Delta, \mathbb{C})$ by

$$A_t(h) = h \circ F_t \quad \text{and} \quad B_t(h) = h \circ G_t, \quad t \geq 0. \quad (18)$$

The operators $\Gamma_f$ and $\Gamma_g$ defined by

$$\Gamma_f(h) = h'f \quad \text{and} \quad \Gamma_g(h) = h'g \quad (19)$$

are their generators, respectively.

Theorem 4. Let $f$ and $g \in \text{Hol}(\Delta, \mathbb{C})$ be generators of one-parameter semigroups $\{F_t\}_{t \geq 0}$ and $\{G_t\}_{t \geq 0}$, respectively. Let $A_t$ and $B_t$ be defined by (18). Then the following are equivalent:

(i) $F_t \circ G_s = G_s \circ F_t$, $s, t \geq 0$, i.e., the semigroups $\{F_t\}_{t \geq 0}$ and $\{G_t\}_{t \geq 0}$ are commuting;
(ii) $A_t \circ B_s = B_s \circ A_t$, $s, t \geq 0$, i.e., the linear semigroups $\{A_t\}_{t \geq 0}$ and $\{B_t\}_{t \geq 0}$ are commuting;
(iii) $\Gamma_f \circ \Gamma_g = \Gamma_g \circ \Gamma_f$, i.e., the linear semigroup generators $\Gamma_f$ and $\Gamma_g$ are commuting;
(iv) the Lie commutator

$$[f, g] = f'g - g'f = 0;$$

(v) $f = \alpha g$ for some $\alpha \in \mathbb{C}$. 

Proof. Suppose that $f \neq 0$. First we prove the equivalence of assertions (i) and (v).

Let (i) holds. If $f(\tau) = 0$, $\tau \in \Delta$, then $\tau$ is the unique common fixed point for the semigroup $\{F_t\}_{t \geq 0}$ generated by $f$, i.e., $F_t(\tau) = \tau$ for all $t \geq 0$ (see, for example, [6], [17]).

If $F_t$ and $G_s$ are commuting for all $s, t \geq 0$, then we have

$$G_s(\tau) = G_s(F_t(\tau)) = F_t(G_s(\tau)).$$

Hence, it follows by the uniqueness of the fixed point $\tau$ that $G_s(\tau) = \tau$ for all $s \geq 0$, and so $g(\tau) = 0$.

Consider the function $h \in \text{Hol}(\Delta, \mathbb{C})$ defined by the differential equation

$$\mu h(z) = h'(z)f(z). \tag{20}$$

It is known that if $\mu = f'(\tau)$ then equation (20) has a unique solution $h \in \text{Hol}(\Delta, \mathbb{C})$ normalized by the condition $h'(\tau) = 1$ (see [17]).

In addition, this function $h$ solves Schroeder’s functional equation

$$h(F_t(z)) = e^{-\mu t}h(z). \tag{21}$$

Now, for any $s, t \geq 0$ we get from (21)

$$h(G_s(F_t(z))) = h(F_t(G_s(z))) = e^{-\mu t}h(G_s(z)).$$

Denote $h_s = h \circ G_s$. Then we have

$$h_s(F_t(z)) = e^{-\mu t}h_s(z). \tag{22}$$

Differentiating (22) at $t = 0^+$ we get

$$\mu h_s(z) = h_s'(z)f(z). \tag{23}$$

Comparing (20) and (23) implies $h_s(z) = \lambda(s)h(z)$ for some $\lambda(s) \in \mathbb{C}$, or

$$h(G_s(z)) = \lambda(s)h(z). \tag{24}$$

Since the left-hand side of the latter equality is differentiable in $s \geq 0$, the scalar function $\lambda(s)$ is differentiable too. Differentiating (24) at $s = 0^+$ we get

$$\lambda'(0)h(z) = -h'(z)g(z). \tag{25}$$

Note that $h(\tau) = 0$ while $h(z) \neq 0$ for all $z \in \Delta, z \neq \tau$. In addition, it can be shown (see [17]) that $h$ is univalent. Hence, $h'(z) \neq 0$ for all $z \in \Delta$.

Finally, we obtain from (20) and (25) that

$$f(z) = \alpha g(z), \quad \text{where} \quad \alpha = -\frac{\mu}{\lambda'(0)}.$$
Now, let us suppose that $f$ has no null point in $\Delta$. Then the function $p : \Delta \rightarrow \mathbb{C}$ given by

$$p(z) = -\int_0^z \frac{ds}{f(s)}$$

is a well defined holomorphic function on $\Delta$ with $p(0) = 0$.

Recall that the semigroup $\{F_t\}_{t \geq 0}$ generated by $f$ can be defined by the Cauchy problem

$$\begin{cases}
\frac{dF_t(z)}{dt} + f(F_t(z)) = 0, & t \geq 0 \\
F_0(z) = z, & z \in \Delta
\end{cases}$$

Substituting here $f(z) = -\frac{1}{p'(z)}$ we obtain

$$p'(F_t(z)) \frac{dF_t(z)}{dt} = dt.$$

Integrating the latter equality on the interval $[0, t]$ we get that $p$ is a solution of Abel’s functional equation

$$p(F_t(z)) = p(z) + t.$$  \hspace{1cm} (28)

Now, for any fixed $s \geq 0$ we have

$$p(G_s(F_t(z))) = p(F_t(G_s(z))) = p(G_s(z)) + t.$$  \hspace{1cm} (29)

Once again, setting $p_s = p \circ G_s$, we have

$$p_s(F_t(z)) = p_s(z) + t.$$  \hspace{1cm} (30)

Differentiating (29) at $t = 0^+$ we get

$$p'_s(z) = -\frac{1}{f(z)},$$  \hspace{1cm} (31)

and by (26), $p_s(z) = p(z) + \kappa(s)$, $\kappa(s) \in \mathbb{C}$, or

$$p'(G_s(z)) = p(z) + \kappa(s).$$  \hspace{1cm} (32)

Differentiating (31) at $s = 0^+$ we obtain the equality

$$p'(z) = -\frac{\kappa'(0)}{g(z)}.$$  \hspace{1cm} (33)

Comparing (30) and (32) gives

$$f = \alpha g \quad \text{with} \quad \alpha = \frac{1}{\kappa'(0)}.$$  \hspace{1cm} (34)

Now we prove that (v) $\Rightarrow$ (i). Let $f = \alpha g$ for some $\alpha \in \mathbb{C}$.

First we assume that $g$ has an interior null-point $\tau \in \Delta$. In this case there is a univalent solution of the differential equation

$$\mu h(z) = h'(z)g(z)$$  \hspace{1cm} (35)
with some \( \mu \in \mathbb{C}, \ \Re \mu \geq 0. \)

Since \( f = \alpha g \), we have that \( h \) is also a solution of the equation

\[
\nu h(z) = h'(z)f(z), \quad \nu = \alpha \mu.
\]

In turn, equations (34) and (35) are equivalent to Schroeder’s functional equations

\[
h (G_s(z)) = e^{-\mu s}h(z), \quad s \geq 0
\]

and

\[
h (F_t(z)) = e^{-\nu t}h(z), \quad t \geq 0, \quad \nu = \alpha \mu,
\]

respectively, where \( \{F_t\}_{t \geq 0} \) is the semigroup generated by \( f \).

Consequently,

\[
F_t (G_s(z)) = h^{-1} (e^{-\nu t}h (G_s(z)))
\]

\[
= h^{-1} (e^{-\nu t} e^{-\mu s}h(z))
\]

\[
= h^{-1} (e^{-\mu s}h (F_t(z))) = G_s (F_t(z))
\]

for all \( s, t \geq 0 \) and we are done.

Now let us assume that \( g \) has a boundary null-point \( \tau \in \partial \Delta \) with \( g'(\tau) \geq 0 \) (see Remark 1 above). In this case for each \( c \in \mathbb{C}, \ c \neq 0, \) Abel’s equations

\[
p (G_s(z)) = p(z) + cs
\]

and

\[
p (F_t(z)) = p(z) + c\alpha t
\]

have the same solution

\[
p(z) = -c \int_0^z \frac{d \varsigma}{g(\varsigma)} = -c\alpha \int_0^z \frac{d \varsigma}{f(\varsigma)}.
\]

which is univalent on \( \Delta \).

Once again we calculate

\[
F_t (G_s(z)) = p^{-1} (p (G_s(z)) + c\alpha t) = p^{-1} (p(z) + c\alpha t + cs)
\]

\[
= p^{-1} (p (F_t(z)) + cs) = G_s (F_t(z)).
\]

The implication (v) \( \Rightarrow \) (i) is proved.

The equivalence of (i) and (ii) is obvious.

To verify the equivalence of (iii) and (iv) we just calculate:

\[
\Gamma_f (\Gamma_g(h)) = h''g f + h'g' f,
\]

\[
\Gamma_g (\Gamma_f(h)) = h''f g + h'f' g.
\]

Hence, \( \Gamma_f \circ \Gamma_g = \Gamma_g \circ \Gamma_f \) if and only if \( f'g - g'f = 0 \).
Now, it is clear, that (v) implies (iv).

Finally we prove the implication (iv) $\Rightarrow$ (v). Obviously, (iv) implies that if $f$ has no null points in $\Delta$ then $g$ also has no null points in $\Delta$ and, hence, (v) follows. If $f(\tau) = 0$ for some $\tau \in \Delta$, then also $g(\tau) = 0$, and by (9) one can write $f(z) = (z - \tau)p(z)$ and $g(z) = (z - \tau)q(z)$, where $p$ and $q$ do not vanish in $\Delta$. Now it follows that

$$[f, g] = (z - \tau)[p, q]$$

Hence, again we have $p = aq$, and hence $f = ag$ for some $a \in \mathbb{C}$, $a \neq 0$.

**Proof of Theorem 3.** First we note, that by Theorem 4

(38) \[ f = \alpha g, \quad \alpha \in \mathbb{C}. \]

(i) Let $f'(\tau) = g'(\tau) = 0$. By Proposition 1 $f$ admits the representation

$$f(z) = (z - \tau)(1 - \tau z)p(z), \quad z \in \Delta,$$

where $p \in \text{Hol}(\Delta, \mathbb{C})$, $\text{Re}p(z) \geq 0$.

Since $f'(\tau) = (1 - |\tau|^2)p(\tau) = 0$, we have $p(\tau) = 0$ and it follows from the maximum principle that $p \equiv 0$. Hence, $f \equiv 0$ and by (38) also $g \equiv 0$.

Assume now $f'(\tau) = g'(\tau) \neq 0$. Then it follows from (38 ) that $\alpha = 1$ and so $f \equiv g$.

(ii) In general, by (38) we have $f^{(k)}(\tau) = \alpha g^{(k)}(\tau)$, $0 < k \leq m$. Hence, the condition $f^{(k)}(\tau) = g^{(k)}(\tau) \neq 0$ for some $0 < k \leq m$ implies that $\alpha = 1$ and, consequently, $f \equiv g$. 

Let $S_f = \{F_t\}_{t \geq 0}$ be the semigroup generated by $f \in \mathcal{G}(\Delta)$. The set $Z(S_f)$ of all semigroups $S = \{G_t\}_{t \geq 0}$ such that

$$F_t \circ G_s = G_s \circ F_t, \quad t, s \geq 0,$$

is called the **centralizer of $S_f$**.

It is clear that for each $f \in \mathcal{G}(\Delta)$ the centralizer $Z(S_f)$ contains $S_{\alpha f}$ for all $\alpha \geq 0$.

Therefore we will say that the centralizer of $S_f$ is **trivial** when the inclusion $S \in Z(S_f)$ implies that $S = S_{\alpha f}$ for some $\alpha \geq 0$.

**Proposition 1.** Let $f$ be the generator of a semigroup $S_f = \{F_t\}_{t \geq 0}$, and let $\tau \in \partial \Delta$ be the Denjoy–Wolff point of $S_f$. Then if one of the following conditions holds then the centralizer $Z(S_f)$ is trivial:

(i) $S_f$ is a hyperbolic type semigroup ($f'(\tau) > 0$) which is not a group;

(ii) $f$ admits the expansion

$$f(z) = a(z - \tau)^3 + o \left( (z - \tau)^3 \right) \quad \text{with} \quad a \neq 0$$

when $z \to \tau$ in each non-tangential approach region at $\tau$. 
The first statement is based on the following simple lemma.

**Lemma 2.** Let $f$ and $g$ be generators of two nontrivial (neither $f$ nor $g$ are identically zero) commuting semigroups $S_f = \{F_t\}_{t \geq 0}$ and $S_g = \{G_t\}_{t \geq 0}$, respectively. Then $S_f$ is of hyperbolic type if and only if $S_g$ is. In this case $f = g$ with real $\alpha$. Moreover, $\alpha < 0$ implies that $S_f$ and $S_g$ are both groups of hyperbolic automorphisms having ‘opposite’ fixed points, i.e., the attractive point for $S_f$ is the repelling point for $S_g$ and conversely.

**Proof.** Since $S_f$ and $S_g$ are commuting then by Theorem 4 there exists $\alpha \in \mathbb{C}$ such that $f = \alpha g$. In our settings $\alpha$ is not zero. If $\tau$ is the Denjoy–Wolff point of $S_f$ then $f(\tau) = 0$ and therefore also $g(\tau) = 0$. Now since $f'(\tau) > 0$ then $g'(\tau) = \frac{1}{\alpha} f'(\tau)$ exists finitely and it must be a real number by Corollary 2. So must be $\alpha$.

Now let us assume that $\alpha$ is negative. Then $g'(\tau) = \frac{1}{\alpha} f'(\tau) < 0$. Hence the semigroup $S_g$ generated by $g$ must have the Denjoy–Wolff point $\sigma \in \overline{\Delta}$ different from $\tau$.

It is clear that $\sigma$ cannot be inside $\Delta$ since otherwise it must be a common fixed point of both semigroups $S_f$ and $S_g$ because of the commuting property.

So, $\sigma \in \partial \Delta$ and $g'(\sigma) \geq 0$ (see [11]), then $f(\sigma) = 0$ and $f'(\sigma) \leq 0$. It follows by a result in [18] that

\begin{equation}
0 < f'(\tau) \leq -f'(\sigma)
\end{equation}

and the equality is possible if and only if $f$ is the generator of a group of hyperbolic automorphisms. From the same theorem we have the reversed inequality for $g$

\begin{equation}
0 \leq g'(\sigma) \leq -g'(\tau)
\end{equation}

that means

\begin{equation}
0 \leq \frac{1}{\alpha} f'(\sigma) \leq -\frac{1}{\alpha} f'(\tau).
\end{equation}

Comparing this inequality with (39) gives us that $f'(\tau) = -f'(\sigma) > 0$ and $g'(\tau) = -g'(\sigma) < 0$ which means that both $f$ and $g$ generate groups of hyperbolic automorphisms with opposite fixed points.

**Remark.** The last assertion of this lemma follows also by a result of Behan (see [5]). Indeed, let $\alpha < 0$. Then the equality $f(z) = \alpha g(z)$ implies that $g'(\tau)$ exists and is a real negative number. So, the Denjoy–Wolff point $\tau$ of the semigroup $S_f$ cannot be the Denjoy–Wolff point of the semigroup $S_g$. Hence by [5] we conclude that $S_f$ and $S_g$ are groups of hyperbolic automorphisms.

**Proof of Proposition 1.** The statement (i) is a direct consequence of the previous lemma. To prove the second statement we note that by Theorem 1 the number $\alpha \tau^2$ is a non-negative real number. On the other hand, since $S_f$ and $S_g$ commute by Theorem 4 there is a number $\alpha \in \mathbb{C}$ such that $f = \alpha g$. 

Therefore, since \( \alpha \neq 0 \) also \( g \) admits the expansion

\[
g(z) = \frac{a}{\alpha} (z - \tau)^3 + o((z - \tau)^3)
\]

and again by Theorem 1 we have that also \( \frac{a}{\alpha} \tau^2 \geq 0 \). This implies that \( \alpha \) is a nonnegative real number. \( \square \)

A natural question which arises in the context of the above theorem is:

- If two elements \( F_{t_0} \) and \( G_{s_0} \) of semigroups \( S_f = \{F_t\}_{t \geq 0} \) and \( S_g = \{G_t\}_{t \geq 0} \) commute for some positive \( t_0 \) and \( s_0 \), do these semigroup \( S_f \) and \( S_g \) commute in the sense:

\[
F_t \circ G_s = G_s \circ F_t
\]

for each pair \( t, s \geq 0 \)?

The answer is immediately affirmative due to a more general result of C. C. Cowen ([9]), Corollary 4.2) if neither \( F_{t_0} \) nor \( G_{s_0} \), respectively, are of parabolic type.

The situation becomes more complicated if \( F_{t_0} \), respectively \( G_{s_0} \), are parabolic.

Example 4.4 in [9] shows that there is a triple of such mappings \( F, G_1 \) and \( G_2 \) in \( \text{Hol}(\Delta) \) for which \( G_1 \) and \( G_2 \) commute with \( F \), but they do not commute with each other.

Nevertheless, under some additional requirements on smoothness at the boundary Denjoy-Wolff point repeating the arguments used in the proof of Theorem 1.2 in [19] one can give an affirmative answer to the above question. Namely,

- Let \( F_{t_0} \) and \( G_{s_0} \) be two commuting elements of semigroups \( S_f \) and \( S_g \), respectively, \( t_0, s_0 > 0 \), and let \( F_{t_0} \) is of parabolic type with a Denjoy–Wolff point \( \tau \in \partial \Delta \). If both \( F_{t_0} \) and \( G_{s_0} \) belong to the class \( C^2(\tau) \) and \( F''_{t_0}(\tau) \) as well as \( G''_{s_0}(\tau) \) do not vanish, then \( f = ag \) for some \( a \in \mathbb{C} \), i.e., the semigroups \( S_f \) and \( S_g \) commute:

\[
F_t \circ G_s = G_s \circ F_t
\]

for all \( t, s \geq 0 \).

REFERENCES


