MULTIPLE POSITIVE SOLUTIONS OF STURM-LIOUVILLE PROBLEMS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to study the existence of multiple positive solutions for the second order Sturm-Liouville problems with impulse effects. The proof is based on the theory of fixed point index in cones.

Keywords: Boundary value problems; Impulse effects; Multiple positive solutions; Fixed point index in cones

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1. INTRODUCTION

This paper is devoted to study the existence of multiple positive solutions for the boundary value problem with impulse effects

\[
\begin{align*}
-Lu & = g(x,u), \quad x \in I', \\
-\Delta(pu')_{x=x_k} & = I_k(u(x_k)), \quad k = 1, 2, \ldots, m, \\
R_1(u) & = \alpha_1 u(0) + \beta_1 u'(0) = 0, \\
R_2(u) & = \alpha_2 u(1) + \beta_2 u'(1) = 0,
\end{align*}
\]

here \(Lu = (p(x)u')' + q(x)u\) is Sturm-Liouville operator, \(I = [0, 1]\), \(I' = I \setminus \{x_1, x_2, \ldots, x_m\}\) and \(0 < x_1 < x_2 < \cdots < x_m < 1\) are given, \(\mathbb{R}^+ = [0, \infty)\), \(g \in \mathbb{C}(I \times \mathbb{R}^+, \mathbb{R}^+)\), \(I_k \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^+)\), \(\Delta(pu')_{x=x_k} = p(x_k)u'(x_k^+) - p(x_k)u'(x_k^-), u'(x_k^+)\) (respectively \(u'(x_k^-)\)) denotes the right limit (respectively left limit) of \(u'(x)\) at \(x = x_k\).

Throughout this paper, we always suppose that

\((S_1)\) \(p(x) \in \mathbb{C}^1([0, 1], \mathbb{R}), p(x) > 0, q(x) \in \mathbb{C}([0, 1], \mathbb{R}), q(x) \leq 0, \alpha_1, \alpha_2, \beta_2 \geq 0, \beta_1 < 0, \alpha_1^2 + \beta_2^2 > 0, \alpha_2^2 + \beta_2^2 > 0.\)

In recent years, second-order differential boundary value problems with impulses have been studied extensively in the literature (see for instance [1, 3, 6, 7, 8, 9, 10, 11]...
and their references). However, most papers are concerned with the case $p(x) = 1$ and $q(x) = 0$. In this paper, we will consider the case $p(x) \neq 1$ and $q(x) \neq 0$. Here we also mention that second order dynamic inclusions on time scales with impulses has been studied in [2].

The existence of positive solutions of problem (1.1) has been studied in [5]. By employing Krasnosel’skii fixed point theorem on compression and expansion of cones, it was proved in [5] that problem (1.1) has at least one positive solution when $g(x, u)$ is either superlinear or sublinear in $u$. Our results in this paper improve those in [5]. The proof is based on fixed point index theory in cones [4].

To conclude the introduction, we introduce the following notation:

\[
g_0 = \liminf_{u \to 0^+} \min_{x \in [0,1]} \frac{g(x, u)}{u}, \quad I_0(k) = \liminf_{u \to 0^+} \frac{I_k(u)}{u},
\]

\[
g_\infty = \liminf_{u \to +\infty} \min_{x \in [0,1]} \frac{g(x, u)}{u}, \quad I_\infty(k) = \liminf_{u \to +\infty} \frac{I_k(u)}{u};
\]

\[
g^\infty = \limsup_{u \to +\infty} \max_{x \in [0,1]} \frac{g(x, u)}{u}, \quad I^\infty(k) = \limsup_{u \to +\infty} \frac{I_k(u)}{u},
\]

\[
g^0 = \limsup_{u \to 0^+} \max_{x \in [0,1]} \frac{g(x, u)}{u}, \quad I^0(k) = \limsup_{u \to 0^+} \frac{I_k(u)}{u}.
\]

Moreover, for the simplicity in the following discussion, we introduce the following hypotheses.

(H1) \( g_0 + \frac{\sigma \sum_{k=1}^{m} I_0(k) \phi_1(x_k)}{\int_0^1 \phi_1(x) dx} > \lambda_1, \quad g_\infty + \frac{\sigma \sum_{k=1}^{m} I_\infty(k) \phi_1(x_k)}{\int_0^1 \phi_1(x) dx} > \lambda_1, \)

(H2) \( g^0 + \frac{\sum_{k=1}^{m} I^0(k) \phi_1(x_k)}{\int_0^1 \frac{m(x) n(x)}{m(1) n(0)} \phi_1(x) dx} < \lambda_1, \quad g^\infty + \frac{\sum_{k=1}^{m} I^\infty(k) \phi_1(x_k)}{\int_0^1 \frac{m(x) n(x)}{m(1) n(0)} \phi_1(x) dx} < \lambda_1, \)

where \( \sigma = \min_{x \in [c_1, c_m]} \min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \) (see section 2), and \( \phi_1(x) \) is the eigenfunction related to the smallest eigenvalue \( \lambda_1 \) of the eigenvalue problem \(-L\phi = \lambda\phi, \ \ R_1(\phi) = R_2(\phi) = 0. \)

(H3) There is a \( p > 0 \) such that \( 0 \leq u \leq p \) and \( 0 \leq x \leq 1 \) implies \( g(x, u) \leq \eta p, \ \ I_k(u) \leq \eta_k p, \)

here \( \eta, \eta_k \geq 0 \) satisfy \( \eta + \sum_{k=1}^{m} \eta_k > 0, \ \ \eta \int_0^1 G(y, y) dy + \sum_{k=1}^{m} G(x_k, x_k) \eta_k < 1 \) and \( G(x, y) \) is the Green’s function of boundary value problem \(-Lu = 0, \ \ R_1(u) = R_2(u) = 0 \) (see section 2).

(H4) There is a \( p > 0 \) such that \( \sigma p \leq u \leq p \) implies \( g(x, u) \geq \lambda p, \ \ 0 \leq x \leq 1, \ \ I_k(u) \geq \lambda_k p, \)
here $\lambda$, $\lambda_k \geq 0$ satisfy $\lambda + \sum_{k=1}^{m} \lambda_k > 0$ and $\lambda \int_{x_1}^{x_m} G(\frac{x}{2}, y) dy + \sum_{k=1}^{m} \lambda_k G(\frac{1}{2}, x_k) > 1$.

2. PRELIMINARIES

In this paper, we shall consider the following space

$$PC(I, \mathbb{R}) = \{u \in \mathbb{C}(I, \mathbb{R}); u'|_{(x_k, x_{k+1})} \in \mathbb{C}(x_k, x_{k+1}),$$

$$u'(x_k^-) = u'(x_k^+), \quad \exists u'(x_k), \quad k = 1, 2, \cdots, m\}$$

with the norm $\|u\|_{PC'} = \max\{\|u\|, \|u'\|\}$, here $\|u\| = \sup_{x \in [0, 1]} |u(x)|$, $\|u'\| = \sup_{x \in [0, 1]} |u'(x)|$.

Then $PC'(I, \mathbb{R})$ is a Banach space.

**Definition 2.1.** A function $u \in PC'(I, \mathbb{R}) \cap C^2(I', \mathbb{R})$ is a solution of (1.1) if it satisfies the differential equation

$$Lu + g(x, u) = 0, \quad x \in I'$$

and the function $u$ satisfies conditions $\Delta (pu')|_{x=x_k} = -I_k(u(x_k))$ and $R_1(u) = R_2(u) = 0$.

Let $Q = I \times I$ and $Q_1 = \{(x, y) \in Q | 0 \leq x \leq y \leq 1\}$, $Q_2 = \{(x, y) \in Q | 0 \leq y \leq x \leq 1\}$. Let $G(x, y)$ is the Green's function of the boundary value problem

$$-Lu = 0, \quad R_1(u) = R_2(u) = 0.$$

Following from [5], $G(x, y)$ can be written by

$$G(x, y) := \begin{cases} \frac{m(x)n(y)}{\omega}, & (x, y) \in Q_1, \\ \frac{m(y)n(x)}{\omega}, & (x, y) \in Q_2. \end{cases}$$

**Lemma 2.2.** [4] Suppose that $(S_1)$ holds, then the Green's function $G(x, y)$, defined by (2.1), possesses the following properties:

(i) $m(x) \in \mathbb{C}^2(I, R)$ is increasing and $m(x) > 0$, $x \in (0, 1]$.

(ii) $n(x) \in \mathbb{C}^2(I, R)$ is decreasing and $n(x) > 0$, $x \in [0, 1)$.

(iii) $(Lm)(x) \equiv 0$, $m(0) = -\beta_1$, $m'(0) = \alpha_1$.

(iv) $(Ln)(x) \equiv 0$, $n(1) = \beta_2$, $n'(1) = -\alpha_2$.

(v) $\omega$ is a positive constant. Moreover, $p(x)(m'(x)n(x) - m(x)n'(x)) \equiv \omega$.

(vi) $G(x, y)$ is continuous and symmetrical over $Q$.

(vii) $G(x, y)$ has continuously partial derivative over $Q_1$, $Q_2$.

(viii) For each fixed $y \in I$, $G(x, y)$ satisfies $LG(x, y) = 0$ for $x \neq y$, $x \in I$. Moreover, $R_1(G) = R_2(G) = 0$ for $y \in (0, 1)$. 


Consider the linear Sturm-Liouville problem

\[-(Lu)(x) = \lambda u(x), \quad R_1(u) = R_2(u) = 0,\]

By the Sturm-Liouville theory of ordinary differential equations (see, for example, [4], [11]), we know that there exists an eigenfunction \(\phi_1(x)\) with respect to the first eigenvalue \(\lambda_1 > 0\) such that \(\phi_1(x) > 0\) for \(x \in (0, 1)\).

Following from Lemma 2.2, it is easy to see that

\[
\min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \frac{m(y)n(y)}{\omega} \leq G(x, y) \leq G(y, y) = \frac{m(y)n(y)}{\omega}, \quad (x, y) \in [0, 1] \times [0, 1].
\]

Let \(E\) be a Banach space and \(K \subset E\) be a closed convex cone in \(E\). For \(r > 0\), let \(K_r = \{u \in K : ||u|| < r\}\) and \(\partial K_r = \{u \in K : ||u|| = r\}\). The following three Lemmas are needed in our argument, which can be found in [4].

**Lemma 2.3.** Let \(\Phi : K \to K\) be a continuous and completely continuous mapping and \(\Phi u \neq u\) for \(u \in \partial K_r\). Then the following conclusions hold:

(i) If \(||u|| \leq ||\Phi u||\) for \(u \in \partial K_r\), then \(i(\Phi, K_r, K) = 0\);

(ii) If \(||u|| \geq ||\Phi u||\) for \(u \in \partial K_r\), then \(i(\Phi, K_r, K) = 1\).

**Lemma 2.4.** Let \(\Phi : K \to K\) be a continuous and completely continuous mapping with \(\mu \Phi u \neq u\) for every \(u \in \partial K_r\) and \(0 < \mu \leq 1\). Then \(i(\Phi, K_r, K) = 1\).

**Lemma 2.5.** Let \(\Phi : K \to K\) be a continuous and completely continuous mapping. Suppose that the following two conditions are satisfied:

(i) \(\inf_{u \in \partial K_r} ||\Phi u|| > 0\); \quad (ii) \(\mu \Phi u \neq u\) for every \(u \in \partial K_r \) and \(\mu \geq 1\).

Then, \(i(\Phi, K_r, K) = 0\).

In applications below, we take \(E = C(I, \mathbb{R})\) and define

\[K = \{u \in C(I, \mathbb{R}) : u(x) \geq \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\} ||u||, x \in I\}.\]

One may readily verify that \(K\) is a cone in \(E\).

Define an operator \(\Phi : K \to K\) by

\[(\Phi u)(x) = \int_0^1 G(x, y)g(y, u(y))dy + \sum_{k=1}^m G(x, x_k)I_k(u(x_k)), \quad x \in I.\]

**Lemma 2.6.** \(\Phi(K) \subset K\). Moreover, \(\Phi : K \to K\) is continuous and completely continuous.
Proof It is easy to see that $\Phi : K \to K$ is continuous and completely continuous. Thus we only need to show $\Phi(K) \subset K$.

In fact, for $u \in K$, by using inequalities (2.2), we have that

$$
\|\Phi u\| \leq \int_0^1 G(y, y)g(y, u(y))dy + \sum_{k=1}^{m} G(x_k, x_k)I_k(u(x_k))
$$

and

$$
(\Phi u)(x) \geq \min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \int_0^1 G(y, y)g(y, u(y))dy \]
+ \min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \sum_{k=1}^{m} G(x_k, x_k)I_k(u(x_k))

\geq \min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \|\Phi u\|, \ x \in [0, 1].
$$

Thus, $\Phi(K) \subset K$. \hfill \Box

Lemma 2.7. If $u$ is a fixed point of the operator $\Phi$, then $u$ is a solution of problem (1.1).

3. MAIN RESULTS

Lemma 3.1. If $(H_3)$ is satisfied, then $i(\Phi, K_p, K) = 1$.

Proof Let $u \in K$ with $\|u\| = p$. It follows from $(H_3)$ that

$$
\|\Phi u\| \leq \int_0^1 G(y, y)g(y, u(y))dy + \sum_{k=1}^{m} G(x_k, x_k)I_k(u(x_k))

\leq p \int_0^1 G(y, y)dy + \sum_{k=1}^{m} G(x_k, x_k)\eta_k < p = \|u\|.
$$

Thus

$$
\|\Phi u\| < \|u\|, \ \forall \ u \in \partial K_p.
$$

It is obvious that $\Phi u \neq u$ for $u \in \partial K_p$. Therefore, $i(\Phi, K_p, K) = 1$, here we use Lemma 2.3. \hfill \Box

Lemma 3.2. If $(H_4)$ is satisfied, then $i(\Phi, K_p, K) = 0$.

Proof Let $u \in K$ with $\|u\| = p$, then

$$
u(x) \geq \min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \|u\| \geq \min_{x \in [x_1, x_m]} \min \left\{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \right\} \|u\| = \sigma p, \ x \in [x_1, x_m].
$$
It follows from (H_4) that
\[
(\Phi u)(\frac{1}{2}) \geq \int_{x_1}^{x_m} G(\frac{1}{2}, y)g(y, u(y))dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_k(u(x_k)) \\
\geq p[\lambda \int_{x_1}^{x_m} G(\frac{1}{2}, y)dy + \sum_{k=1}^{m} \lambda_k G(\frac{1}{2}, x_k)] \\
> p = \|u\|.
\]
Therefore,
\[
\|\Phi u\| > \|u\|, \quad \forall \; u \in \partial K_p.
\]
Clearly \( \Phi u \neq u \) for \( u \in \partial K_p \). So, \( i(\Phi, K_p, K) = 0 \), here we use Lemma 2.3. \( \square \)

**Theorem 3.3.** Assume that (H_1) and (H_3) are satisfied. Then problem (1.1) has at least two positive solutions \( u_1 \) and \( u_2 \) with
\[
0 < \|u_1\| < p < \|u_2\|.
\]

**Proof** According to Lemma 3.1, we have that
\[
(3.1) \quad i(\Phi, K_p, K) = 1.
\]
Since (H_1) holds, then there exists \( 0 < \varepsilon < 1 \) such that
\[
(3.2) \quad (1 - \varepsilon)[g_0 + \frac{\sigma \sum_{k=1}^{m} I_0(k)\phi_1(x_k)}{\int_{0}^{1} \phi_1(x)dx}] > \lambda_1, \quad (1 - \varepsilon)[g_{\infty} + \frac{\sigma \sum_{k=1}^{m} I_{\infty}(k)\phi_1(x_k)}{\int_{0}^{1} \phi_1(x)dx}] > \lambda_1.
\]
By the definitions of \( g_0, \; I_0, \) one can find \( 0 < r_0 < p \) such that
\[
g(x, u) \geq g_0(1 - \varepsilon)u, \quad I_k(u) \geq I_0(k)(1 - \varepsilon)u, \quad \forall \; x \in [0, 1], 0 < u < r_0.
\]
Let \( r \in (0, r_0) \), then for \( u \in \partial K_r, \; x \in [x_1, x_m] \), we have
\[
u(x) \geq \min_{x \in [x_1, x_m]} \min \{ \frac{m(x)}{m(1)}, \frac{n(x)}{n(0)} \}\|u\| = \sigma r.
\]
Thus
\[
(\Phi u)(\frac{1}{2}) = \int_{0}^{1} G(\frac{1}{2}, y)g(y, u(y))dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_k(u(x_k)) \\
\geq \int_{x_1}^{x_m} G(\frac{1}{2}, y)g(y, u(y))dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_k(u(x_k)) \\
\geq g_0(1 - \varepsilon) \int_{x_1}^{x_m} G(\frac{1}{2}, y)u(y)dy + (1 - \varepsilon) \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_0(k)u(x_k) \\
\geq (1 - \varepsilon)\sigma r[g_0 \int_{x_1}^{x_m} G(\frac{1}{2}, y)dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k)I_0(k)],
\]
from which we see that \( \inf_{u \in \partial K_r} \|\Phi u\| > 0 \), namely, hypothesis (i) of Lemma 2.5 holds.
Next we show that \( \mu \Phi u \neq u \) for any \( u \in \partial K_r \) and \( \mu \geq 1 \).
If this is not true, then there exist \( u_0 \in \partial K_r \) and \( \mu_0 \geq 1 \) such that \( \mu_0 \Phi u_0 = u_0 \).

Note that \( u_0(x) \) satisfies

\[
\begin{aligned}
Lu_0(x) + \mu_0 q(x, u_0(x)) &= 0, \quad x \in I', \\
-\Delta (p u_0')_{x=x_k} &= \mu_0 I_k(u_0(x_k)), \quad k = 1, 2, \ldots, m, \\
\alpha_1 u_0(0) + \beta_1 u_0'(0) &= 0 \\
\alpha_2 u_0(1) + \beta_2 u_0'(1) &= 0.
\end{aligned}
\]

Multiply equation (3.3) by \( \phi_1(x) \) and integrate from 0 to 1, note that

\[
\begin{aligned}
&\int_0^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx = \int_0^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
&+ \sum_{k=1}^{m-1} \int_{x_k}^{x_{k+1}} \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
&+ \int_{x_m}^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx \\
= &\phi_1(x_1)p(x_1)u_0'(x_1 - 0) - \phi_1(0)p(0)u_0'(0) - \int_0^{x_1} p(x)u_0'(x)\phi_1(x)dx \\
&+ \int_0^{x_1} q(x)u_0(x)\phi_1(x)dx + \sum_{k=1}^{m-1} [\phi_1(x_{k+1})p(x_{k+1})u_0'(x_{k+1} - 0) \\
&- \phi_1(x_k)p(x_k)u_0'(x_k + 0) - \int_{x_k}^{x_{k+1}} p(x)u_0'(x)\phi_1(x)dx \\
&+ \int_{x_k}^{x_{k+1}} q(x)u_0(x)\phi_1(x)dx + \phi_1(1)p(1)u_0'(1) - \phi_1(x_m)p(x_m)u_0'(x_m + 0) \\
&- \int_{x_m}^1 p(x)u_0'(x)\phi_1(x)dx + \int_{x_m}^1 q(x)u_0(x)\phi_1(x)dx \\
= &- \sum_{k=1}^{m} \Delta (p(x_k)u_0'(x_k))\phi_1(x_k) - \int_0^1 p(x)\phi_1'(x)u_0'(x)dx + \int_0^1 q(x)\phi_1(x)u_0(x)dx \\
&+ \phi_1(1)p(1)u_0'(1) - \phi_1(0)p(0)u_0'(0).
\end{aligned}
\]

Also note that

\[
\begin{aligned}
\int_0^1 p(x)\phi_1'(x)u_0'(x)dx &= \int_0^1 p(x)\phi_1'(x)du_0(x) \\
&= p(1)\phi_1'(1)u_0(1) - p(0)\phi_1'(0)u_0(0) - \int_0^1 u_0(x)(p(x)\phi_1'(x))'dx \\
&= p(1)\phi_1'(1)u_0(1) - p(0)\phi_1'(0)u_0(0) + \int_0^1 u_0(x)q(x)\phi_1(x)dx \\
&+ \lambda_1 \int_0^1 u_0(x)\phi_1(x)dx.
\end{aligned}
\]
Thus, by the boundary conditions, we have
\[
\int_0^1 \phi_1(x)[(p(x)u_0'(x))' + q(x)u_0(x)]dx = -\sum_{k=1}^m \Delta(p(x_k)u_0'(x_k))\phi_1(x_k)
- p(1)\phi_1'(1)u_0(1) + p(0)\phi_1'(0)u_0(0)
- \int_0^1 u_0(x)q(x)\phi_1(x)dx - \lambda_1 \int_0^1 u_0(x)\phi_1(x)dx
+ \int_0^1 q(x)\phi_1(x)u_0(x)dx + \phi_1(1)p(1)u_0'(1) - \phi_1(0)p(0)u_0'(0)
\]
\[
= -\sum_{k=1}^m \Delta(p(x_k)u_0'(x_k))\phi_1(x_k) - \lambda_1 \int_0^1 u_0(x)\phi_1(x)dx
+ \sum_{k=1}^m \mu_k I_k(u_0(x_k))\phi_1(x_k) - \lambda_1 \int_0^1 u_0(x)\phi_1(x)dx.
\]

So we obtain
\[
\lambda_1 \int_0^1 u_0(x)\phi_1(x)dx = \mu_0 \sum_{k=1}^m I_k(u_0(x_k))\phi_1(x_k) + \mu_0 \int_0^1 \phi_1(x)g(x, u_0(x))dx
\geq (1 - \varepsilon) \sum_{k=1}^m I_0(k)\phi_1(x_k)u_0(x_k) + (1 - \varepsilon)g_0 \int_0^1 \phi_1(x)u_0(x)dx.
\]

Since \(u_0(x) \geq \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}\|u_0\| \geq \min\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\}r\), we have \(\int_0^1 \phi_1(x)u_0(x)dx > 0\), and so from the above inequality we see that \(\lambda_1 \geq (1 - \varepsilon)g_0\). If \(\lambda_1 = (1 - \varepsilon)g_0\), then \(I_0(k) = 0, k = 1, 2, \ldots, m\). But from (3.2) we have \((1 - \varepsilon)g_0 > \lambda_1\), which is a contradiction. So \(\lambda_1 > (1 - \varepsilon)g_0\). Thus
\[
[\lambda_1 - (1 - \varepsilon)g_0] \int_0^1 u_0(x)\phi_1(x)dx \geq (1 - \varepsilon) \sum_{k=1}^m I_0(k)\phi_1(x_k)u(x_k)
\geq (1 - \varepsilon)\sigma r \sum_{k=1}^m I_0(k)\phi_1(x_k).
\]

Since \(\int_0^1 u_0(x)\phi_1(x)dx \leq r \int_0^1 \phi_1(x)dx\), we have
\[
[\lambda_1 - (1 - \varepsilon)g_0] \int_0^1 \phi_1(x)dx \geq (1 - \varepsilon)\sigma \sum_{k=1}^m I_0(k)\phi_1(x_k),
\]
which contradicts (3.2) again. Hence \(\Phi\) satisfies the hypotheses of Lemma 2.5 in \(K_r\). Thus
\[
(3.4) \quad i(\Phi, K_r, K) = 0.
\]

On the other hand, from (H1), there exists \(H > p\) such that
\[
(3.5) \quad g(x, u) \geq g_\infty(1 - \varepsilon)u, \quad I_k(u) \geq I_\infty(k)(1 - \varepsilon)u, \forall x \in [0, 1], \ u \geq H.
\]
Let \( C = \max_{0 \leq u \leq R} \max_{0 \leq x \leq 1} |g(x, u) - g_\infty(1 - \varepsilon)u| + \sum_{k=1}^{m} \max_{0 \leq u \leq H} |I_k(u) - I_\infty(k)(1 - \varepsilon)u| \). It is clear that

\[
(3.6) \quad g(x, u) \geq g_\infty(1 - \varepsilon)u - C, \quad I_k(u) \geq I_\infty(k)(1 - \varepsilon)u - C, \quad \forall \ x \in [0, 1], \ u \geq 0.
\]

Choose \( R > R_0 := \max\{H, \sigma, p\} \) and let \( u \in \partial K_R \). Since \( u(x) \geq \sigma ||u|| = \sigma R > H \) for \( x \in [x_1, x_m] \), from (3.5) we see that

\[
g(x, u(x)) \geq g_\infty(1 - \varepsilon)u(x) \geq \sigma g_\infty(1 - \varepsilon)R, \quad \forall \ x \in [x_1, x_m].
\]

\[
I_k(u(x_k)) \geq \sigma I_\infty(k)(1 - \varepsilon)R.
\]

Essentially the same reasoning as above yields \( \inf_{u \in \partial K_R} ||\Phi u|| > 0 \). Next we show that if \( R \) is large enough, then \( \mu \Phi u \neq u \) for any \( u \in \partial K_R \) and \( \mu \geq 1 \). In fact, if there exist \( u_0 \in \partial K_R \) and \( \mu_0 \geq 1 \) such that \( \mu_0 \Phi u_0 = u_0 \), then \( u_0(x) \) satisfies equation (3.3).

Multiply equation (3.3) by \( \phi_1(x) \) and integrate from 0 to 1, using integration by parts in the left side to obtain

\[
\begin{align*}
\lambda_1 \int_0^1 u_0(x)\phi_1(x)dx &= \mu_0 \sum_{k=1}^{m} I_k(u_0(x_k))\phi_1(x_k) + \mu_0 \int_0^1 g(x, u_0(x))\phi_1(x)dx \\
&\geq (1 - \varepsilon) \sum_{k=1}^{m} I_\infty(k)\phi_1(x_k)u_0(x_k) + (1 - \varepsilon)g_\infty \int_0^1 u_0(x)\phi_1(x)dx \\
&- C\left(\sum_{k=1}^{m} \phi_1(x_k) + \int_0^1 \phi_1(x)dx\right).
\end{align*}
\]

If \( g_\infty \leq \lambda_1 \), then we have

\[
[\lambda_1 - (1 - \varepsilon)g_\infty] \int_0^1 u_0(x)\phi_1(x)dx + C \left(\sum_{k=1}^{m} \phi_1(x_k) + \int_0^1 \phi_1(x)dx\right) \\
\geq (1 - \varepsilon) \sum_{k=1}^{m} I_\infty(k)\phi_1(x_k)u_0(x_k),
\]

thus

\[
||u_0||[\lambda_1 - (1 - \varepsilon)g_\infty] \int_0^1 \phi_1(x)dx + C \left(\sum_{k=1}^{m} \phi_1(x_k) + \int_0^1 \phi_1(x)dx\right) \\
\geq (1 - \varepsilon)\sigma ||u_0|| \sum_{k=1}^{m} I_\infty(k)\phi_1(x_k)
\]

and

\[
(3.7a) \quad ||u_0|| \leq \frac{C\left(\sum_{k=1}^{m} \phi_1(x_k) + \int_0^1 \phi_1(x)dx\right)}{(1 - \varepsilon)\sigma \sum_{k=1}^{m} I_\infty(k)\phi_1(x_k) - [\lambda_1 - (1 - \varepsilon)g_\infty] \int_0^1 \phi_1(x)dx} =: \bar{R}.
\]
If \( g_\infty > \lambda_1 \), we can choose \( \varepsilon > 0 \) such that \( (1 - \varepsilon)g_\infty > \lambda_1 \), then we have

\[
C \left( \sum_{k=1}^{m} \phi_1(x_k) + \int_{0}^{1} \phi_1(x)dx \right) \geq [(1 - \varepsilon)g_\infty - \lambda_1] \int_{0}^{1} \phi_1(x)u_0(x)dx \\
\geq [(1 - \varepsilon)g_\infty - \lambda_1] \|u_0\| \int_{0}^{1} \left( \frac{m(x) n(x)}{m(1) n(0)} \right) \phi_1(x)dx.
\]

Thus

\[
(3.7b) \quad \|u_0\| \leq \frac{C \left( \sum_{k=1}^{m} \phi_1(x_k) + \int_{0}^{1} \phi_1(x)dx \right)}{[(1 - \varepsilon)g_\infty - \lambda_1] \int_{0}^{1} \left( \frac{m(x) n(x)}{m(1) n(0)} \right) \phi_1(x)dx} =: \tilde{R}.
\]

Let \( R > \max\{p, \tilde{R}\} \), then for any \( u \in \partial K_R \) and \( \mu \geq 1 \), we have \( \mu \Phi u \neq u \). Hence hypothesis (ii) of Lemma 2.5 is satisfied and

\[
(3.8) \quad i(\Phi, K_R, K) = 0.
\]

In view of (3.1), (3.4) and (3.8), we obtain

\[
i(\Phi, K_R \setminus \bar{K}_p, K) = -1, \ i(\Phi, K_p \setminus \bar{K}_r, K) = 1.
\]

Then \( \Phi \) has fixed points \( u_1 \) and \( u_2 \) in \( K_p \setminus \bar{K}_r \) and \( K_R \setminus \bar{K}_p \), respectively, which means \( u_1(x) \) and \( u_2(x) \) are positive solution of the problem (1.1) and \( 0 < \|u_1\| < p < \|u_2\| \).

**Corollary 3.4.** The conclusion of Theorem 3.3 is valid if (H1) is replaced by

(H1\*) \( g_0 = \infty \) or \( \sum_{k=1}^{m} I_0(k)\phi_1(x_k) = \infty; \quad g_\infty = \infty \) or \( \sum_{k=1}^{m} I_\infty(k)\phi_1(x_k) = \infty. \)

**Theorem 3.5.** Assume that (H2) and (H4) are satisfied, then problem (1.1) has at least two positive solutions \( u_1 \) and \( u_2 \) with

\[
0 < \|u_1\| < p < \|u_2\|.
\]

**Proof** According to Lemma 3.2, we have that

\[
(3.9) \quad i(\Phi, K_p, K) = 0.
\]

Since (H2) holds, there exists \( 0 < \varepsilon < \min\{\lambda_1 - g^0, \lambda_1 - g^\infty\} \) such that

\[
(3.10) \quad (\lambda_1 - \varepsilon - g^0) \int_{0}^{1} \left( \frac{m(x) n(x)}{m(1) n(0)} \right) \phi_1(x)dx > \sum_{k=1}^{m} (I^0(k) + \varepsilon)\phi_1(x_k),
\]

and

\[
(3.11) \quad (\lambda_1 - \varepsilon - g^\infty) \int_{0}^{1} \left( \frac{m(x) n(x)}{m(1) n(0)} \right) \phi_1(x)dx > \sum_{k=1}^{m} (I^\infty(k) + \varepsilon)\phi_1(x_k).
\]

One can find \( 0 < r_0 < p \) such that

\[
(3.12) \quad g(x, u) \leq (g^0 + \varepsilon)u, \quad I_k(u) \leq (I^0(k) + \varepsilon)u, \forall x \in [0, 1], \ 0 \leq u \leq r_0.
\]
Let $r \in (0, r_0)$. Now we prove that $\mu \Phi u \neq u$ for any $x \in \partial K_r$ and $0 < \mu \leq 1$. If this is not true, then there exist $u_0 \in \partial K_r$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \Phi u_0 = u_0$. Then $u_0(x)$ satisfies equation (3.3). Multiply equation (3.3) by $\phi_1(x)$ and integrate from 0 to 1, using (3.12), to obtain

$$
\lambda_1 \int_0^1 u_0(x)\phi_1(x)dx = \mu_0 \sum_{k=1}^m I_k(u_0(x_k))\phi_1(x_k) + \mu_0 \int_0^1 \phi_1(x)g(x, u_0(x))dx \\
\leq \sum_{k=1}^m (I^0(k) + \varepsilon)u_0(x_k)\phi_1(x_k) + \int_0^1 \phi_1(x)u_0(x)dx(g^0 + \varepsilon),
$$

i.e.

$$(\lambda_1 - g^0 - \varepsilon) \int_0^1 u_0(x)\phi_1(x)dx \leq \sum_{k=1}^m (I^0(k) + \varepsilon)u_0(x_k)\phi_1(x_k) \leq r \sum_{k=1}^m (I^0(k) + \varepsilon)\phi_1(x_k).$$

Since $u_0(x) \geq \min\left\{\frac{m(x)}{m(1)}, \frac{n(x)}{n(0)}\right\} ||u_0|| \geq \left(\frac{m(x)}{m(1)} \frac{n(x)}{n(0)}\right) r$, and so from the above inequality we see that

$$(\lambda_1 - g^0 - \varepsilon) \int_0^1 \left(\frac{m(x)}{m(1)} \frac{n(x)}{n(0)}\right) \phi_1(x)dx \leq \sum_{k=1}^m (I^0(k) + \varepsilon)\phi_1(x_k),$$

which is a contradiction. By Lemma 2.4, we have

$$i(\Phi, K_r, K) = 1.$$  

On the other hand, from (H2), there exist $H > p$ such that

$$g(x, u) \leq (g^\infty + \varepsilon)u, \; I_k(u) \leq (I^\infty(k) + \varepsilon)u \; \forall \; x \in [0, 1], \; u \geq H.$$  

Let $C = \max_{0 \leq u \leq H} \max_{0 \leq x \leq 1} |g(x, u) - (g^\infty + \varepsilon)u| + \sum_{k=1}^m \max_{0 \leq u \leq H} |I_k(u) - (I^\infty(k) + \varepsilon)u|$. It is clear that

$$g(x, u) \leq (g^\infty + \varepsilon)u + C, \; I_k(u) \leq (I^\infty(k) + \varepsilon)u + C, \; \forall \; x \in [0, 1], \; u \geq 0.$$  

Next we show that if $R$ is large enough, then $\mu \Phi u \neq u$ for any $u \in \partial K_R$ and $0 < \mu \leq 1$. In fact, if there exist $u_0 \in \partial K_R$ and $0 < \mu_0 \leq 1$ such that $\mu_0 \Phi u_0 = u_0$, then $u_0(x)$ satisfies equation (3.3). Multiply equation (3.3) by $\phi_1(x)$ and integrate from 0 to 1, using (3.14), to obtain

$$\lambda_1 \int_0^1 u_0(x)\phi_1(x)dx = \mu_0 \sum_{k=1}^m I_k(u_0(x_k))\phi_1(x_k) + \mu_0 \int_0^1 g(x, u_0(x))\phi_1(x)dx \\
\leq \sum_{k=1}^m (I^\infty(k) + \varepsilon)\phi_1(x_k)u_0(x_k) + \int_0^1 \phi_1(x)u_0(x)dx(g^\infty + \varepsilon) + C \sum_{k=1}^m \phi_1(x_k) + \int_0^1 \phi_1(x)dx,$$

i.e.,
\[(\lambda_1 - g^{\infty} - \varepsilon) \int_0^1 u_0(x) \phi_1(x) dx \leq \sum_{k=1}^m (I_{(k)}^\infty + \varepsilon) \phi_1(x_k)u_0(x_k) + C\left(\sum_{k=1}^m \phi_1(x_k) + \int_0^1 \phi_1(x) dx\right)\]

(3.15)

Also we have \(\int_0^1 u_0(x) \phi_1(x) dx \geq \|u_0\| \int_0^1 (\frac{m(x)}{m(\sigma(x))}) \phi_1(x) dx\), and this together with (3.15) yields

\[
\|u_0\| \leq \frac{C\left(\sum_{k=1}^m \phi_1(x_k) + \int_0^1 \phi_1(x) dx\right)}{(\lambda_1 - g^{\infty} - \varepsilon) \int_0^1 (\frac{m(x)}{m(\sigma(x))}) \phi_1(x) dx - \sum_{k=1}^m (I_{(k)}^\infty + \varepsilon) \phi_1(x_k)} =: \bar{R}.
\]

Let \(R = \max\{p, \bar{R}\}\), then for any \(x \in \partial K_R\) and \(0 < \mu \leq 1\), we have \(\mu \Phi u \neq u\). Thus

(3.16)

\(i(\Phi, K_R, K) = 1\).

In view of (3.9), (3.13) and (3.16), we obtain

\(i(\Phi, K_R \setminus \bar{K}_p, K) = 1, \quad i(\Phi, K_p \setminus \bar{K}_r, K) = -1\).

Then \(\Phi\) has fixed points \(u_1\) and \(u_2\) in \(K_p \setminus \bar{K}_r\) and \(K_R \setminus \bar{K}_p\), respectively, which means \(u_1(x)\) and \(u_2(x)\) are positive solution of problem (1.1) and \(0 < \|u_1\| < p < \|u_2\|\).

**Corollary 3.6.** The conclusion of Theorem 3.5 is valid if (H2) is replaced by

(H2*) \(g^0 = 0\) and \(I^0(k) = 0, \quad k = 1, 2, \ldots, m\); \(g^{\infty} = 0\) and \(I^\infty(k) = 0, \quad k = 1, 2, \ldots, m\).

The proof of the following two Theorems follows the ideas in the proof of Theorems 3.3 and 3.5. Here we omit it here.

**Theorem 3.7.** Assume the following condition is satisfied:

\[
g_0 + \frac{\sigma \sum_{k=1}^m I_0(k) \phi_1(x_k)}{\int_0^1 \phi_1(x) dx} > \lambda_1, \quad g^{\infty} + \frac{\sum_{k=1}^m I_{(k)}^\infty \phi_1(x_k)}{\int_0^1 (\frac{m(x)}{m(\sigma(x))}) \phi_1(x) dx} < \lambda_1.
\]

Then (1.1) has at least one positive solution.

**Corollary 3.8.** Assume the following condition is satisfied:

\(g_0 = \infty\) or \(\sum_{k=1}^m I_0(k) \phi_1(x_k) = \infty, \quad g^{\infty} = 0\) and \(I^\infty(k) = 0, \quad k = 1, \ldots, m\)

Then (1.1) has at least one positive solution.
Theorem 3.9. Assume the following condition is satisfied:

\[
g^0 + \sum_{k=1}^{m} \frac{I^0(k)\phi_1(x_k)}{\int_0^1 \frac{m(x)n(x)}{m(1)n(0)}\phi_1(x)dx} < \lambda_1, \quad g_\infty + \frac{\sigma \sum_{k=1}^{m} I^\infty(k)\phi_1(x_k)}{\int_0^1 \phi_1(x)dx} > \lambda_1.
\]

Then (1.1) has at least one positive solution.

Corollary 3.10. Assume that

\[
g^0 = 0 \quad \text{and} \quad I^0(k) = 0, \quad k = 1, \ldots, m; \quad g_\infty = \infty \quad \text{or} \quad \sum_{k=1}^{m} I^\infty(k)\phi_1(x_k) = \infty.
\]

Then (1.1) has at least one positive solution.

Example 3.11. Consider the following impulsive boundary value problem

\[
Lu + Au^\alpha + Bu^\beta = 0, \quad x \in I', \quad 0 < \alpha < 1 < \beta, \quad A > 0, \quad B > 0,
\]

\[
-\Delta(pu')|_{x=x_k} = c_k u(x_k), \quad c_k \geq 0,
\]

\[
R_1(u) = \alpha_1 u(0) + \beta_1 u'(0) = 0,
\]

\[
R_2(u) = \alpha_2 u(1) + \beta_2 u'(1) = 0,
\]

here \(Lu = (p(x)u')' + q(x)u\). Assume that \((S_1)\) is satisfied. Then problem (3.17) has at least two positive solutions \(u_1\) and \(u_2\) with

\[
0 < ||u_1|| < 1 < ||u_2||
\]

provided

\[
A + B < \frac{1}{d}(1 - \sum_{k=1}^{m} G(x_k, x_k)c_k), \quad d = \int_0^1 G(y,y)dy.
\]

Proof To see this we will apply Theorem 3.3 (or Corollary 3.4).

By (3.18), \(\eta > 0\) is chosen such that

\[
A + B < \eta < \frac{1}{d}(1 - \sum_{k=1}^{m} G(x_k, x_k)c_k).
\]

Set

\[
g(x, u) = Au^\alpha + Bu^\beta.
\]

Note

\[
g_0 = \infty, \quad g_\infty = \infty,
\]

so \((H_1)\) (or \((H_1^*)\)) holds.

Let \(\eta_k = c_k\), then \(\eta, \ \eta_k\) satisfy

\[
\eta \int_0^1 G(y,y)dy + \sum_{k=1}^{m} G(x_k, x_k)\eta_k < 1.
\]

Let \(p = 1\), then for \(0 \leq u \leq p\), we have

\[
g(x, u) = Au^\alpha + Bu^\beta \leq A + B < \eta p = \eta,
\]
and
\[ I_k(u) = c_k u = \eta_k u \leq \eta_k p, \]
thus (H_3) holds. The result now follows from Theorem 3.3 (or Corollary 3.4). □

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