EXISTENCE OF SOLUTIONS OF MULTIPOINT BOUNDARY VALUE PROBLEMS FOR A SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT. Assuming the uniqueness of an n-point boundary value problem, for some $n \geq 4$ for

$$y'' = f(x, y, y')$$

existence of unique solutions are proved for all $n \geq 2$.

1. INTRODUCTION

We are concerned with the question of the existence of a unique solution of the n-point boundary value problem for the second order differential equation

$$y'' = f(x, y, y')$$

$$y(x_1) = y_1; y(x_n) - \sum_{i=2}^{n-1} y(x_i) = y_2.$$  

We assume throughout:

(A): $f : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

(B): Solutions of initial value problems for (1) are unique and exist on all of $(a, b)$.

(C): For a given $n \geq 2$ and any points $a < x_1 < x_2 < \cdots < x_n < b$, if $y$ and $z$ are solutions of (1) such that

$$y(x_1) = z(x_1); y(x_n) - \sum_{i=2}^{n-1} y(x_i) = z(x_n) - \sum_{i=2}^{n-1} z(x_i)$$

then $y(x) = z(x)$ for all $a < x < b$.

There has been much interest in multipoint boundary value problems for the second order differential equation. See [1] and the references contained therein.

The main theorem of [1] is the following

**Theorem 1.** Assume that conditions (A), (B) and (C) hold. Then for each $2 \leq k \leq n$, there is a unique solution of (1) that satisfies (2).
2. MAIN RESULT

Our purpose is to complement Theorem 1 with the following

**Theorem 2.** Assume conditions (A), (B) and that (C) holds for some \( n \geq 4 \). Then for each \( n \geq 2 \), there is a unique solution of (1) that satisfies (2).

**Proof.** Suppose (C) holds for \( k = n \geq 4 \). Assume that it does not hold for \( k + 1 \), i.e. there are two distinct solutions \( u \) and \( v \) of (1) that satisfy (2) when \( n = k + 1 \).

Define \( g := u - v \). Then

\[
g(x_1) = 0
\]
\[
g(x_{k+1}) = \sum_{i=2}^{k} g(x_i).
\]

Note that \( g(x) \neq 0 \) for \( x > x_1 \), otherwise \( u \) and \( v \) satisfy (C) for \( n = 2 \). In that case, it follows from Theorem 1 that \( u \equiv v \). Also see [2].

Next note that \( g(t) \neq g(s) \) for \( t \neq s \) and both \( t \) and \( s \) greater than \( x_1 \), otherwise \( g(x_1) = 0 \) and \( g(s) = g(t) \) so that \( u \) and \( v \) satisfy (C) for \( n = 3 \). Again, Theorem 1 implies \( u \equiv v \).

Thus, we can assume without loss of generality that \( g(x) \) is positive and monotone increasing for \( x > x_1 \). Since \( g \) is also continuous, it follows that there exist points \( t_1 < t_2 \) in \((x_1, x_2)\) so that

\[
g(x_2) = g(t_1) + g(t_2).
\]

In that case \( u \) and \( v \) satisfy (C) for \( n = 4 \) and consequently applying Theorem 1 again, \( u \equiv v \).

The following example shows that for \( n = 3 \) (C) can hold while it does not hold for \( n = 4 \).

**Example 1.** There are examples of (1) with solutions that satisfy (2) for \( n = 3 \) but do not have solutions that satisfy (2) if \( n = 4 \).

Consider the linear equation

\[
y'' + py' + qy = f.
\]

Every solution of (4) is of the form

\[
y = c_1 y_1 + c_2 y_2 + z
\]

where \( z \) is any solution of (4) and \( y_1 \) and \( y_2 \) are linearly independent solutions of

\[
y'' + py' + qy = 0.
\]

Suppose \( a < x_1 < x_2 < x_3 < b \). We want a solution of (1) that satisfies

\[
y(x_1) = \alpha; \ y(x_3) - y(x_2) = \beta
\]
Choose \( y_1 \) so that \( y_1(x_1) = 0 \). Since \( y_1 \) and \( y_2 \) are linearly independent, \( y_2(x_1) \neq 0 \). It follows that \( c_2 = \frac{\alpha - z(x_1)}{y_2(x_1)} \). Now \( \beta = y(x_3) - y(x_2) = c_1[y_1(x_3) - y_1(x_2)] + c_2[y_2(x_3) - y_2(x_2)] + [z(x_3) - z(x_2)] \). Therefore, (4) has a solution which is unique satisfying (6) provided \( y_1(x_3) - y_1(x_2) \neq 0 \). This is the case when \( y_1 \) is increasing on the interval \((a, b)\).

In the case of the four point problem

\[
y(x_1) = \alpha; \quad y(x_4) - y(x_3) - y(x_2) = \beta
\]

with \( a < x_1 < x_2 < x_3 < x_4 < b \), (4) has a solution satisfying (7), provided \( y_1(x_4) - y_1(x_3) - y_1(x_2) \neq 0 \), which fails to hold for all points in \((a, b)\).

**Example 2.** There are examples of (1) with solutions that satisfy (2) for \( n = 2 \) but do not have solutions that satisfy (2) if \( n = 3 \).

As in Example 1, there is a solution of (4) that satisfies

\[
y(x_1) = \alpha; \quad y(x_2) = \beta
\]

provided \( y_1(x_2) \neq 0 \), however unless \( y_1 \) is monotone on \((a, b)\) there is not always a solution of (4) that satisfies (6).

**Acknowledgement.** The author is indebted to the referee for valuable help with this paper.

**REFERENCES**
