POSITIVE SOLUTIONS OF SINGULAR BOUNDARY VALUE PROBLEMS FOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. Existence criteria are established for positive solutions of singular boundary value problems for nonlinear second order ordinary and delay differential equations. Here the nonlinearities may be singular at phase variables and positive solutions ‘pass through’ the singularities.

Keywords and Phrases. Singular boundary value problem, existence, positive solution, Vitali’s convergence theorem.

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1. INTRODUCTION

Let $a$, $b$, $c$ and $T$ be positive numbers, $b < a$, $c < a$ and $J = [0,T]$. Let $\alpha_i \in C^1(J)$, $i = 1, 2$, satisfy the following conditions:

(j) $\alpha_i(t) < t$ and $\alpha_i'(t) \geq \Delta_i > 0$ for $t \in J$,

(jj) there exist $\tau_i \in (0,T)$ such that $\alpha_i(\tau_i) = 0$.

Set $r = \min \{\alpha_1(0), \alpha_2(0)\} (< 0)$. Let $\varphi \in C^0([r,0])$, $\varphi(0) = a$, $0 < \varphi(t) < a$ for $t \in [r,0)$ and $\varphi(\alpha_2(t)) > c$ for $t \in [0,\tau_2]$. Consider the singular boundary value problem (BVP)

(1) $x''(t) = \mu q(t) \left( f_1(t,x(t),x(\alpha_1(t))) + f_2(t,x(t),x(\alpha_2(t))) \right)$,

(2) $x(t) = \varphi(t)$ for $t \in [r,0]$, $x(T) = 0$,

where $\mu \geq 0$ is a constant, $q(t) > 0$ for $t \in (0,T)$, $f_1(t,x,y) \geq 0$ for $(t,x,y) \in J \times (0,b) \cup (b,a) \times (0,a)$ and $f_2(t,x,y) \geq 0$ for $(t,x,y) \in J \times (0,a) \times [0,c) \cup (c,a)$.

The function $f_1$ (resp. $f_2$) may be singular at the points $x = 0$, $x = b$ and $x = a$ (resp. $x = 0$ and $x = a$) of the phase variable $x$ and at the points $y = 0$ and $y = a$ (resp. $y = c$ and $y = a$) of the phase variable $y$.

Set

$\mathcal{E} = \{x : x \in C^0([r,T]) \cap C^1(J), x \text{ is decreasing on } J\}.$

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We say that \( x \) is a solution of BVP (1), (2) in the set \( E \) if \( x \in E \cap C^2(J_1) \), where \( J_1 \) denotes the interval \((0, T)\) with the exception of at most two points if \( f_1 \) is singular at the point \( x = b \) of the phase variable \( x \) and \( f_2 \) is singular at the point \( y = c \) of the phase variable \( y \), \( x \) fulfills the boundary conditions (2) and (1) is satisfied on \( J_1 \).

If (1) is independent of the delays \( \alpha_1 \) and \( \alpha_2 \), we obtain a singular BVP of the type
\[
\begin{align*}
x''(t) &= \mu q(t) f_*(t, x(t)), \\
x(0) &= a, \quad x(T) = 0
\end{align*}
\]
where \( f_* \geq 0 \) may be singular at \( x = 0 \), \( x = b \) and \( x = a \) of the phase variable \( x \).

This problem has been considered for instance in [3], [6] and [22] but here \( f_* \) may be singular only at \( x = 0 \).

Many existence results have been established for boundary value problems with second–order functional differential equations with delay which have no singularities in the phase variables; see, e.g., [2–5,8,11,13,14,18–21] and their references. Boundary value problems for second order functional differential equations with singularities in the phase variable having positive solutions have been considered in [1], [7] and [16].

For example in [1] is discussed BVP
\[
\begin{align*}
x''(t) + q(t) \tilde{f}(t, x(t - r)) &= 0, \quad t \in (0, 1) \setminus \{r\}, \\
x(t) &= \mu(t) \text{ for } t \in [-r, 0], \quad x(1) = 0
\end{align*}
\]
where \( 0 < r < 1 \) is a constant, \( \mu(0) = 0, \mu > 0 \) on \([-r, 0)\) and \( \tilde{f} \geq 0 \) on \((0, 1) \times (0, \infty)\) may by singular at \( x = 0 \) of the phase variable \( x \). Here the singularity of \( \tilde{f} \) at \( x = 0 \) ‘appears’ in positive solutions only at the fixed point \( t = 0 \) and \( t = 1 \) where solutions vanishing. In our paper the singularities of \( f_1 \) and \( f_2 \) ‘appear’ in positive solutions of BVP (1), (2) not only at the fixed point \( t = 0 \) and \( t = T \), but positive solutions ‘pass through’ the singularities of \( f_1 \) and \( f_2 \) in inner points of \((0, T)\) (if \( f_1 \) is singular at \( x = b \) and \( f_2 \) is singular at \( y = c \)).

The aim of this paper is to give conditions for the existence of a solution of BVP (1), (2) in the set \( E \). Our results are proved by the regularity and sequential techniques. First, we construct 2–parameter family of auxiliary regular BVPs \((8)_n, (9)_n\) depending on parameters \((\lambda, n) \in [0, 1] \times \mathbb{N}\) and obtain a priori bounds for their solutions (Lemma 1). Then applying the topological transversality theorem (see, e.g., [9], [10]), we prove the existence of positive solutions of the auxiliary regular BVPs (Lemma 2). The main result for the original BVP (1), (2) (Theorem 1) follows from Arzelà–Ascoli’s theorem and a modification of Vitali’s convergence theorem (see, e.g., [12], [17]) given in Lemma 3. Finally, Corollary 1 generalized results obtained in [3], [6] and [22].

Throughout the paper we will use the following assumptions:

\((H_1)\) \( q \in C^0((0, T)), q > 0 \) on \((0, T)\) and \( Q = \sup\{q(t) : t \in J\} < \infty; \)
(H_2) \ f_1 : J \times D_1 \times (0, a) \to [0, \infty) \text{ is continuous with } D_1 = (0, b) \cup (b, a), \ f_1(t, x, y) \leq g_1(x) + h_1(y), \ (t, x, y) \in J \times D_1 \times (0, a), \text{ where } g_1 \geq 0 \text{ is continuous on } D_1, \ h_1 \geq 0 \text{ is continuous on } (0, a) \text{ and } \int_0^a (g_1(s) + h_1(s)) \, ds < \infty, \int_0^a h_1(\varphi(\alpha_1(t))) \, dt < \infty;

(H_3) \ f_2 : J \times (0, a) \times D_2 \to [0, \infty) \text{ is continuous with } D_2 = [0, c) \cup (c, a), \ f_2(t, x, y) \leq g_2(x) + h_2(y), \ (t, x, y) \in J \times (0, a) \times D_2, \text{ where } g_2 \geq 0 \text{ is continuous on } (0, a), \ h_2 \geq 0 \text{ is continuous on } D_2 \text{ and } \int_0^a (g_2(s) + h_2(s)) \, ds < \infty, \int_0^a h_2(\varphi(\alpha_2(t))) \, dt < \infty;

(H_4) \ h_1 \text{ and } h_2 \text{ are nondecreasing on } (a - \varepsilon_0, a) \text{ for an } \varepsilon_0 > 0;

(H_5) \lim_{v \to 0} v g_1(b + v) = 0, \lim_{v \to 0} v h_2(c + v) = 0.

2. NOTATION AND LEMMAS

Let assumptions (H_1)-(H_3) be satisfied. Let \( n_* \in \mathbb{N} \),

\[ n_* > 2 \max \left\{ \frac{1}{b}, \frac{1}{c}, \frac{1}{a-b}, \frac{1}{a-c}, \frac{1}{2\varepsilon_0} \right\} \]

and \( \mathbb{N}_* = \{ n : n \in \mathbb{N}, n \geq n_* \} \). By Urysohn’s lemma, for each \( n \in \mathbb{N}_* \), there exists \( p_n \in C^0(J \times \mathbb{R}) \) such that \( 0 \leq p_n(t, x) \leq 1 \) for \( t \in J \times \mathbb{R} \), \( p_n(t, x) = 0 \) for \( (t, x) \in J \times (-\infty, 1/(2n)] \) and \( p_n(t, x) = 1 \) for \( (t, x) \in J \times [1/n, \infty) \). Let \( \varphi_n \in C^0([r, 0]), \xi_n \in C^0(\mathbb{R}), f_{1n}, f_{2n} \in C^0(J \times \mathbb{R}^2), g_{1n} \in C^0((0, a]), h_{2n} \in C^0([0, a]) \) and \( l_{1n}, l_{2n} \in \mathbb{R} \) be defined for \( n \in \mathbb{N}_* \) by the formulas

\[ \varphi_n(t) = \begin{cases} a - (1/n) & \text{if } \varphi(t) \geq a - (1/n) \\ \varphi(t) & \text{if } \varphi(t) < a - (1/n), \end{cases} \]

\[ \xi_n(x) = \begin{cases} a - (1/n) & \text{for } x \geq a - (1/n) \\ x & \text{for } 1/(2n) \leq x < a - (1/n) \\ 1/(2n) & \text{for } x < 1/(2n), \end{cases} \]

\[ f_{1n}(t, x, y) = \begin{cases} f_1(t, a - (1/n), \xi_n(y)) & \text{for } (t, x, y) \in J \times [a - (1/n), \infty) \times \mathbb{R} \\ f_1(t, x, \xi_n(y)) & \text{for } (t, x, y) \in J \times (b + (1/n), a - (1/n)) \times \mathbb{R} \\ f_1(t, x, \xi_n(y))(x - b + (1/n)) & \text{for } (t, x, y) \in J \times [b - (1/n), b + (1/n)] \times \mathbb{R} \\ f_1(t, x, \xi_n(y))(x - b + (1/n)) & \text{for } (t, x, y) \in J \times [1/(2n), b - (1/n)] \times \mathbb{R} \\ x - (1/2n) & \text{for } (t, x, y) \in J \times (-\infty, 1/(2n)) \times \mathbb{R}, \end{cases} \]
Consider the family of the auxiliary regular BVPs
\[
\begin{aligned}
f_{2n}(t, x, y) &= \begin{cases}
f_2(t, \xi_n(x), a - (1/n)) & \text{for } (t, x, y) \in J \times \mathbb{R} \times [a - (1/n), \infty) \\
f_2(t, \xi_n(x), y) & \text{for } (t, x, y) \in J \times \mathbb{R} \times (c + (1/n), a - (1/n)) \\
(n/2)\left( f_2(t, \xi_n(x), c + (1/n))(y - c + (1/n)) - f_2(t, \xi_n(x), c - (1/n))(y - c - (1/n)) \right) & \text{for } (t, x, y) \in J \times \mathbb{R} \times [c - (1/n), c + (1/n)] \\
p_n(t, x)p_n(t, y)f_2(t, \xi_n(x), y) & \text{for } (t, x, y) \in J \times \mathbb{R} \times (\infty, c - (1/n)),
\end{cases}
\end{aligned}
\]

By \((H_2), (H_3)\) and \((H_5)\), \(\lim_{n \to \infty} l_{1n} = \lim_{n \to \infty} l_{2n} = 0, \)
\[
(5) \quad f_{1n}(t, x, y) \leq g_{1n}(x) + h_1(y), \quad f_{2n}(t, x, y) \leq g_2(x) + h_{2n}(y)
\]
for \((t, x, y) \in J \times [1/(2n), a - (1/n)] \times [1/(2n), a - (1/n)]\). In addition,
\[
\int_u^v g_{1n}(s) \, ds \leq \int_u^v g_1(s) \, ds + nl_{1n} \min \left\{ \frac{2}{n}, v - u \right\},
\]
\[
\int_u^v h_{2n}(s) \, ds \leq \int_u^v h_2(s) \, ds + nl_{2n} \min \left\{ \frac{2}{n}, v - u \right\}
\]
and
\[
\int_S g_{1n}(s) \, ds \leq \int_S g_1(s) \, ds + nl_{1n} \min \left\{ \frac{2}{n}, m(S) \right\},
\]
\[
\int_S h_{2n}(s) \, ds \leq \int_S h_2(s) \, ds + nl_{2n} \min \left\{ \frac{2}{n}, m(S) \right\}
\]
for \(n \in \mathbb{N}_+, 0 \leq u \leq v \leq a\) and any measurable \(S \subset [0, a]\), where \(m(S)\) stands for the Lebesgue measure of \(S\).

Consider the family of the auxiliary regular BVPs
\[
(8)_{n\lambda} \quad x''(t) = \lambda \mu q(t) \left( f_{1n}(t, x(t), x(\alpha_1(t))) + f_{2n}(t, x(t), x(\alpha_2(t))) \right),
\]
\[
(9)_n \quad x(t) = \varphi_n(t) \quad \text{for } t \in [r, 0], \quad x(T) = \frac{1}{2n}
\]
depending on the parameters $\lambda \in [0,1]$ and $n \in \mathbb{N}_*$. 

We say that $x \in C^0([r,T]) \cap C^1(J) \cap C^2((0,T))$ is a solution of BVP (8)$_{n\lambda}$, (9)$_n$ if $x$ fulfills the boundary conditions (9)$_n$ and (8)$_{n\lambda}$ is satisfied for $t \in (0,T)$.

Set
\begin{equation}
A_1 = \mu Q \left( \int_0^{r_1} h_1(\varphi(\alpha_1(t))) \, dt + \int_0^{r_2} h_2(\varphi(\alpha_2(t))) \, dt \right),
\end{equation}
\begin{equation}
B_1 = 2\mu Q \left( \int_0^a (g_1(s) + g_2(s)) \, ds + \frac{1}{\Delta_1} \int_0^a h_1(s) \, ds \right.
+ \frac{1}{\Delta_2} \int_0^a h_2(s) \, ds + 2 \sup \left\{ b_1n + \frac{b_2n}{\Delta_2} : n \in \mathbb{N}_* \right\}
\end{equation}
and
\begin{equation}
K = A_1 + \sqrt{A_1^2 + B_1 + \left( \frac{a}{T} \right)^2}.
\end{equation}

**Lemma 1.** Let assumptions (H$_1$)–(H$_5$) be satisfied, $(\lambda, n) \in [0,1] \times \mathbb{N}_*$. Let $x$ be a solution of BVP (8)$_{n\lambda}$, (9)$_n$. Then
\begin{equation}
\frac{1}{2n} \leq x(t) \leq a - \frac{1}{n},
\end{equation}
\begin{equation}
0 \leq -x'(t) \leq K
\end{equation}
for $t \in J$, where $K$ is defined by (12).

**Proof.** If $\lambda \mu = 0$ then
\begin{equation*}
x(t) = \begin{cases}
\varphi_n(t) & \text{for } t \in [r,0) \\
3 - 2na - \frac{2na}{2n} & \text{for } t \in J
\end{cases}
\end{equation*}
and $x$ satisfies (13) and (14). Let $\lambda \mu > 0$. Since $x'' \geq 0$ on $(0,T)$, $x'$ is nondecreasing on $J$ and $x(0) = a - (1/n) > 0$, $x(T) = 1/(2n) < x(0)$ imply $x'(0) < 0$. Hence $x(t) \leq a - (1/n)$ for $t \in J$. If $\min\{x(t) : t \in J\} = x(t_0) < 1/(2n)$ then $t_0 \in (0,T)$, $x'(t_0) = 0$ and
\begin{equation*}
x''(t_0) = \lambda \mu q(t_0) \left( f_{1n}(t_0, x(t_0), x(\alpha_1(t_0))) + f_{2n}(t_0, x(t_0), x(\alpha_2(t_0))) \right)
= \lambda \mu q(t_0) \left( x(t_0) - \frac{1}{2n} \right) < 0,
\end{equation*}
which is impossible. Hence $x(t) \geq 1/(2n)$ for $t \in J$. In addition, $x' \leq 0$ on $J$ since from $x'(\nu) > 0$ for some $\nu \in (0,T)$ we have $x' > 0$ on $[\nu, T]$ which implies $x(T) > 1/(2n)$, a contradiction. We have proved that $x$ satisfies inequalities (13) and $x' \leq 0$ on $J$.

We are going to show that $-x'(t) \leq K$ for $t \in J$. Since $x'' \geq 0$ on $(0,T)$ and $x' \leq 0$ on $J$, we have $x'(0) \leq x'(t) \leq x'(T)$ for $t \in J$ and from $x(0) = a - (1/n)$,
\[ x(T) = 1/(2n) \text{ it follows that } x'(T) \geq (1/T)(3/(2n) - a) > -a/T. \] By (5), (13) and (H3),

\[
x''(t) = \lambda \mu q(t) \left( f_{1n}(t, x(t), x(\alpha_1(t))) + f_{2n}(t, x(t), x(\alpha_2(t))) \right)
\]

\[
\leq \mu Q \left( g_{1n}(x(t)) + h_1(x(\alpha_1(t))) + g_2(x(t)) + h_{2n}(x(\alpha_2(t))) \right)
\]

for \( t \in (0, T) \). Whence

\( 2x''(t)x'(t) \geq 2\mu Q \left( g_{1n}(x(t)) + h_1(x(\alpha_1(t))) + g_2(x(t)) + h_{2n}(x(\alpha_2(t))) \right)x'(t) \)

for \( t \in (0, T) \). By condition (j), \( 1/\alpha_i(t) \leq 1/\Delta_i, \) \( t \in J, i = 1, 2 \) and from (j), (jj) and (H4) we obtain \( x'(\alpha_i(t)) \leq x'(t), \) \( t \in [\tau_i, T], \) and \( h_i(\phi_n(\alpha_i(t))) \leq h_i(\phi(\alpha_i(t))), \) \( t \in [0, \tau_i] \) for \( i = 1, 2 \) and \( n \in \mathbb{N}_* \). Consequently (cf. (6)),

\[
\int_0^T g_{1n}(x(t))x'(t) \, dt = \int_0^{x(T)} g_{1n}(s) \, ds = \int_{a-(1/n)}^{1/(2n)} g_{1n}(s) \, ds
\]

\[
\geq - \int_0^a g_{1n}(s) \, ds \geq - \int_0^a g_1(s) \, ds - 2l_{1n},
\]

\[
\int_0^T h_1(x(\alpha_1(t)))x'(t) \, dt = \int_0^{\tau_1} h_1(x(\alpha_1(t)))x'(t) \, dt + \int_{\tau_1}^T h_1(x(\alpha_1(t)))x'(t) \, dt
\]

\[
\geq x'(0) \int_0^{\tau_1} h_1(\phi_n(\alpha_1(t))) \, dt + \int_{\tau_1}^T \frac{h_1(x(\alpha_1(t)))x(\alpha_1(t))'}{\alpha_1'(t)} \, dt
\]

\[
\geq x'(0) \int_0^{\tau_1} h_1(\phi(\alpha_1(t))) \, dt + \frac{1}{\Delta_1} \int_{x(0)}^{x(\alpha_1(T))} h_1(s) \, ds
\]

\[
\geq x'(0) \int_0^{\tau_1} h_1(\phi(\alpha_1(t))) \, dt - \frac{1}{\Delta_1} \int_0^a h_1(s) \, ds,
\]

\[
\int_0^T g_2(x(t))x'(t) \, dt = \int_{x(0)}^{x(T)} g_2(s) \, ds \geq - \int_0^a g_2(s) \, ds
\]

and

\[
\int_0^T h_{2n}(x(\alpha_2(t)))x'(t) \, dt = \int_0^{\tau_2} h_{2n}(x(\alpha_2(t)))x'(t) \, dt + \int_{\tau_2}^T h_{2n}(x(\alpha_2(t)))x'(t) \, dt
\]

\[
\geq x'(0) \int_0^{\tau_2} h_2(\phi_n(\alpha_2(t))) \, dt + \int_{\tau_2}^T \frac{h_{2n}(x(\alpha_2(t)))x(\alpha_2(t))'}{\alpha_2'(t)} \, dt
\]

\[
\geq x'(0) \int_0^{\tau_2} h_2(\phi(\alpha_2(t))) \, dt + \frac{1}{\Delta_2} \int_{x(0)}^{x(\alpha_2(T))} h_{2n}(s) \, ds
\]

\[
\geq x'(0) \int_0^{\tau_2} h_2(\phi(\alpha_2(t))) \, dt - \frac{1}{\Delta_2} \left( \int_0^a h_2(s) \, ds + 2l_{2n} \right).
\]

Integrating (15) from 0 to \( T \) and using the above inequalities, we see that

\[
(x'(T))^2 - (x'(0))^2 \geq 2A_1 x'(0) - B_1
\]
where $A_1$ and $B_1$ are defined by (10) and (11) respectively. Then
\[-x'(0) \leq A_1 + \sqrt{A_1^2 + B_1 + (x'(T))^2} \leq A_1 + \sqrt{A_1^2 + B_1 + \left(\frac{a}{T}\right)^2} = K\]
since $(x'(T))^2 \leq (a/T)^2$.

Lemma 2. Let assumptions $(H_1)$–$(H_5)$ be satisfied and $n \in \mathbb{N}_+$. Then there exists a solution $x$ of BVP $(8)_n$, $(9)_n$ satisfying inequalities (13) and (14) for $t \in J$, where $K$ is defined by (12).

Proof. Set
\[\mathcal{U} = \{x : x \in C^0([r, T]), \ x(t) = \varphi_n(t) \text{ for } t \in [r, 0]\}\]
and
\[\mathcal{K} = \{x : x \in \mathcal{U}, \ 0 < x(t) < a \text{ for } t \in J\}.\]
Then $\mathcal{U}$ is a convex subset of the Banach space $C^0([r, T])$ equipped with the sup–norm and $\mathcal{K}$ is open in $\mathcal{U}$. Let $\overline{\mathcal{K}}$ and $\partial \mathcal{K}$ denote, respectively, the closure and the boundary of $\mathcal{K}$ in $\mathcal{U}$. Define the operator
\[\Lambda : [0, 1] \times \overline{\mathcal{K}} \to C^0([r, T])\]
by
\[\Lambda(\lambda, x)(t) = \varphi_n(t) \quad \text{for } t \in [r, 0),\]
\[\Lambda(\lambda, x)(t) = a - \frac{1}{n} + \frac{3 - 2na}{2nT}t - \frac{\lambda t}{T} \int_0^T (T - s)q(s) \left( f_{1n}(s, x(s), x(\alpha_1(s))) + f_{2n}(s, x(s), x(\alpha_2(s))) \right) ds + \lambda \mu \int_0^t (t - s)q(s) \left( f_{1n}(s, x(s), x(\alpha_1(s))) + f_{2n}(s, x(s), x(\alpha_2(s))) \right) ds\]
for $t \in [0, T]$. Obviously, $\Lambda$ is a compact operator. Suppose that $\Lambda(\lambda_0, x_0) = x_0$ for some $(\lambda_0, x_0) \in [0, 1] \times \partial \mathcal{K}$. Then $x_0$ is a solution of BVP $(8)_{n\lambda_0}$, $(9)_n$, and so $1/(2n) \leq x_0(t) \leq a - (1/n)$ for $t \in J$ by Lemma 1. Hence $x_0 \notin \partial \mathcal{K}$, a contradiction. Therefore $\Lambda(\lambda, x) \neq x$ for $(\lambda, x) \in [0, 1] \times \partial \mathcal{K}$. Since for $x \in \overline{\mathcal{K}}$ we have
\[\Lambda(0, x)(t) = \begin{cases} 
\varphi_n(t) & \text{for } t \in [r, 0) \\
a - \frac{1}{n} + \frac{3 - 2na}{2nT}t & \text{for } t \in J,
\end{cases}\]
$\Lambda(0, \cdot)$ is a constant operator and $\Lambda(0, \cdot) \in \mathcal{K}$. By the topological transversality theorem, there exists a fixed point $x$ of the operator $\Lambda(1, \cdot)$. Clearly, $x$ is a solution of BVP $(8)_{n1}$, $(9)_n$ and, by Lemma 1, (13) and (14) hold. 

\[\square\]
Theorem 1. Let \( \{w_n(t)\} \subset L_1([\alpha, \beta]) \) be a sequence of nonnegative functions on \([\alpha, \beta]\) converging in measure on \([\alpha, \beta]\) to \(w(t)\). Let \( \{p_n(t)\} \subset L_1([\alpha, \beta]) \) and \( w_n(t) \leq p_n(t) \) for a.e. \( t \in [\alpha, \beta] \) and each \( n \in \mathbb{N} \). Suppose that for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that for each at most countable set pairwise disjoint intervals \((u_i, v_i) \subset [\alpha, \beta], i \in I\) with \( \sum_{i \in I} (v_i - u_i) < \delta \) we have \( \sum_{i \in I} \int_{u_i}^{v_i} p_n(s) \, ds < \varepsilon \) for \( n \in \mathbb{N} \). Then \( w \in L_1([\alpha, \beta]) \) and

\[
\lim_{n \to \infty} \int_{\alpha}^{\beta} w_n(t) \, dt = \int_{\alpha}^{\beta} w(t) \, dt.
\]

Proof. To prove the assertion of the lemma we use Vitali’s convergence theorem. Fix \( \varepsilon > 0 \) and let \( \delta > 0 \) be from the assumption. Let \( \mathcal{M} \subset [\alpha, \beta] \) be a measurable set, \( m(\mathcal{M}) < \delta/2 \). Then there exists an open set \( \mathcal{M}_1 \subset [\alpha, \beta], \mathcal{M} \cap (\alpha, \beta) \subset \mathcal{M}_1 \) such that \( m(\mathcal{M}_1) < \delta \). As \( \mathcal{M}_1 \) is open bounded, \( \mathcal{M}_1 \) is a union of at most countable set of intervals \((\alpha_i, \beta_i), i \in I_\varepsilon \) without common elements, \( \mathcal{M}_1 = \bigcup_{i \in I_\varepsilon} (\alpha_i, \beta_i) \). Then \( \sum_{i \in I_\varepsilon} \int_{\alpha_i}^{\beta_i} p_n(t) \, dt < \varepsilon \) and

\[
\int_{\mathcal{M}} w_n(t) \, dt < \int_{\mathcal{M}} p_n(t) \, dt < \int_{\mathcal{M}_1} p_n(t) \, dt = \sum_{i \in I_\varepsilon} \int_{\alpha_i}^{\beta_i} p_n(t) \, dt < \varepsilon
\]

for \( n \in \mathbb{N} \). We have proved that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( 0 \leq \int_{\mathcal{M}} w_n(t) \, dt < \varepsilon \) for each \( n \in \mathbb{N} \) and any measurable \( \mathcal{M} \subset [\alpha, \beta], m(\mathcal{M}) < \delta/2 \). Hence \( \{w_n(t)\} \) has uniformly absolutely continuous integrals on \([\alpha, \beta]\) and, by Vitali’s convergence theorem, \( w \in L_1([\alpha, \beta]) \) and (16) holds. \( \square \)

3. EXISTENCE RESULTS AND AN EXAMPLE

Theorem 1. Let assumptions \((H_1)–(H_5)\) be satisfied. Set

\[
\mu_T = \frac{a^2}{(2aA + BT)T}
\]

where

\[
A = Q \left( \int_0^{\tau_1} h_1(\varphi(\alpha_1(t))) \, dt + \int_0^{\tau_2} h_2(\varphi(\alpha_2(t))) \, dt \right)
\]

and

\[
B = 2Q \left( \int_0^a (g_1(s) + g_2(s)) \, ds + \frac{1}{\Delta_1} \int_0^a h_1(s) \, ds + \frac{1}{\Delta_2} \int_0^a h_2(s) \, ds \right).
\]

If \( 0 \leq \mu \leq \mu_T \) then \( BVP \) (1), (2) has a solution in the set \( \mathcal{E} \).

Proof. Fix \( \mu \in [0, \mu_T] \). By Lemma 2, for each \( n \in \mathbb{N}_* \), there exists a solution \( x_n \) of \( BVP \) \((8)_{n1}, (9)_n\) such that \( x_n(t) = \varphi_n(t) \) for \( t \in [r, 0] \) and

\[
1/2n \leq x_n(t) \leq a - (1/n), \quad 0 \leq -x_n'(t) \leq K
\]

for \( t \in J \), where \( K > 0 \) is defined by (12). Since \( |\varphi_n(t) - \varphi(t)| \leq 1/n \) for \( t \in [r, 0] \) and \( n \in \mathbb{N}_* \), there is no loss of generality in assuming \( \{x_n(t)\}_{n \in \mathbb{N}_*} \) is uniformly convergent.
on \([r, T]\), and let \(\lim_{n \to \infty} x_n(t) = x(t)\). Then \(x \in C^0([r, T])\), \(x(t) = \varphi(t)\) for \(t \in [r, 0]\), \(x(0) = a\), \(x(T) = 0\) and \(x\) is nondecreasing on \(J\). Hence there exists \(\gamma \in (0, T]\) such that \(a \geq x(t) > 0\) for \(t \in [0, \gamma]\) and \(x(t) = 0\) for \(t \in [\gamma, T]\). We now show that \(x\) is decreasing on \([0, \gamma]\). If not, there exist \(0 \leq t_1 < t_2 < \gamma\) such that \(x(t) = x(t_1)\) for \(t \in [t_1, t_2]\). From \(x''_n \geq 0\) on \((0, T)\) and the equalities

\[
x_n(t_2) - x_n(t_1) = x_n'(\zeta)(t_2 - t_1),
\]

\[
x_n(t_2) - 1/(2n) = x_n(t_2) - x_n(T) = x_n'(\tau)(t_2 - T),
\]

where \(t_1 < \zeta < t_2 < \tau < T\), we obtain

\[
x_n'(\zeta) \leq x_n'(\tau) = \frac{2nx_n(t_2) - 1}{2n(t_2 - T)}.
\]

Then

\[
x_n(t_2) - x_n(t_1) \leq \frac{2nx_n(t_2) - 1}{2n(t_2 - T)}(t_2 - t_1), \quad n \in \mathbb{N}_*
\]

and

\[
\lim_{n \to \infty} (x_n(t_2) - x_n(t_1)) \leq \lim_{n \to \infty} \frac{2nx_n(t_2) - 1}{2n(t_2 - T)}(t_2 - t_1) = \frac{x(t_2)}{t_2 - T}(t_2 - t_1) < 0,
\]

contrary to \(\lim_{n \to \infty}(x_n(t_2) - x_n(t_1)) = x(t_2) - x(t_1) = 0\). Hence

\[
\lim_{n \to \infty} \left( f_{1n}(t, x_n(t), x_n(\alpha_1(t))) + f_{2n}(t, x_n(t), x_n(\alpha_2(t))) \right) = f_1(t, x(t), x(\alpha_1(t))) + f_2(t, x(t), x(\alpha_2(t)))
\]

(21)

for a.e. \(t \in [0, \gamma]\).

Fix \(t_\ast \in (0, \gamma)\). We are going to show that there exist \(\Phi < 0\) and \(n_1 \in \mathbb{N}_*\) such that

\[
x_n'(t) \leq \Phi \quad \text{for } t \in [0, t_\ast], \quad n \geq n_1.
\]

(22)

Assume, on the contrary, that there exists a subsequence \(\{k_n\}\) of \(\mathbb{N}_*\) such that \(\lim_{n \to \infty} x_{k_n}'(t_\ast) = 0\). From

\[
x_{k_n}(t_\ast) - 1/(2k_n) = x_{k_n}(t_\ast) - x_{k_n}(T) = x_{k_n}'(\delta_n)(t_\ast - T) \leq x_{k_n}'(t_\ast)(t_\ast - T),
\]

where \(\delta_n \in (t_\ast, T)\), it follows that

\[
\lim_{n \to \infty} (x_{k_n}(t_\ast) - 1/(2k_n)) \leq (t_\ast - T) \lim_{n \to \infty} x_{k_n}'(t_\ast) = 0,
\]

contrary to \(\lim_{n \to \infty}(x_{k_n}(t_\ast) - 1/(2k_n)) = x(t_\ast) > 0\). In order to prove that

\[
f_1(t, x(t), x(\alpha_1(t))) + f_2(t, x(t), x(\alpha_2(t))) \in L_1([0, t_\ast])
\]

and

\[
\lim_{n \to \infty} \int_0^{t_\ast} \left( f_{1n}(s, x_n(s), x_n(\alpha_1(s))) + f_{2n}(s, x_n(s), x_n(\alpha_2(s))) \right) ds
\]

(23)

\[
= \int_0^{t_\ast} \left( f_1(s, x(s), x(\alpha_1(s))) + f_2(s, x(s), x(\alpha_2(s))) \right) ds
\]
we use Lemma 3. Let \( \varepsilon > 0 \). Then there exists \( n_2 \in \mathbb{N} \), \( n_2 \geq n_1 \), such that

\[
l_{1n} < \frac{\varepsilon |\Phi|}{14}, \quad l_{2n} < \frac{\varepsilon |\Delta_2|\Phi|}{14}
\]

for \( n \geq n_2 \) and there exists \( \delta_0 > 0 \) such that (cf. \((H_2)\) and \((H_3)\))

\[
\begin{align*}
\int_{D_0} (g_1(s) + g_2(s)) \, ds &< \frac{\varepsilon |\Phi|}{7}, \quad \int_{D_1} h_1(s) \, ds < \frac{\varepsilon |\Delta_1|\Phi|}{7}, \\
\int_{D_2} h_2(s) \, ds &< \frac{\varepsilon |\Delta_2|\Phi|}{7}, \quad \int_{D \cap [0, \tau_1]} h_1(\varphi_1(t)) \, dt < \frac{\varepsilon |\Phi|}{7K}
\end{align*}
\]

and

\[
\int_{D \cap [0, \tau_2]} h_2(\varphi_2(t)) \, dt < \frac{\varepsilon |\Phi|}{7K}.
\]

for any measurable subsets \( D_j \) of \([0, a] \) \((j = 0, 1, 2)\) and measurable \( D \subset J \) such that \( m(D_0) < \delta_0 \), \( m(D_j) < \delta_0 \max\{\alpha_j'(t) : t \in J\} \) \((j = 1, 2)\) and \( m(D) < \delta_0/K \). We recall that \( \tau_1 \) and \( \tau_2 \) are numbers from condition \((j)\). Let

\[
\delta = \frac{1}{K} \min \left\{ \delta_0, \frac{\varepsilon |\Phi|}{\max\{m l_{1n} : n_1 \leq n \leq n_2\}}, \frac{\varepsilon |\Delta_2|\Phi|}{7 \max\{m l_{2n} : n_1 \leq n \leq n_2\} \max\{\alpha_2'(t) : t \in J\}} \right\}
\]

Let \( \{(u_i, v_i)\}_{i \in I_1} \) be at most countable sequence of pairwise disjoint intervals \((u_i, v_i) \subset [0, t_0] \) such that \( \sum_{i \in I_1} (v_i - u_i) < \delta \) and set \( M = \sum_{i \in I_1} (u_i, v_i) \). Let \( i \in I_1 \) and \( n \geq n_1 \). Then (cf. \((22)\) and \((H_4)\))

\[
|\Phi| \int_{u_i}^{v_i} \left( g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_{2n}(x_n(t)) + h_2n(x_n(\alpha_2(t))) \right) \, dt
\]

\[
\leq \left| \int_{u_i}^{v_i} \left( g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_{2n}(x_n(t)) + h_2n(x_n(\alpha_2(t))) \right) x_n'(t) \, dt \right|
\]

\[
\leq \int_{x_n(u_i)}^{x_n(v_i)} (g_1(s) + g_2(s)) \, ds + H_1(u_i, v_i) + H_2(u_i, v_i)
\]

where

\[
H_1(u_i, v_i) \leq \begin{cases} 
K \int_{u_i}^{v_i} h_1(\varphi(\alpha_1(t))) \, dt & \text{if } v_i \leq \tau_1 \\
K \int_{u_i}^{v_i} h_1(\varphi(\alpha_1(t))) \, dt + \frac{1}{\Delta_1} \int_{x_n(\alpha_1(u_i))}^{a} h_1(s) \, ds & \text{if } u_i < \tau_1 < v_i \\
\frac{1}{\Delta_1} \int_{x_n(\alpha_1(v_i))} h_1(s) \, ds & \text{if } u_i \geq \tau_1
\end{cases}
\]
and

\[
H_2(u_i, v_i) \leq \left\{\begin{array}{ll}
K \int_{u_i}^{v_i} h_2(\varphi(\alpha_2(t))) \, dt & \text{if } v_i \leq \tau_2 \\
K \int_{u_i}^{\tau_2} h_2(\varphi(\alpha_2(t))) \, dt + \frac{1}{\Delta_2} \int_{x_n(\alpha_2(v_i))}^{\tau_2} h_2 n(s) \, ds & \text{if } u_i < \tau_2 < v_i \\
\frac{1}{\Delta_2} \int_{x_n(\alpha_2(v_i))}^{\tau_2} h_2 n(s) \, ds & \text{if } u_i \geq \tau_2.
\end{array}\right.
\]

Since \( |x_n(v_i) - x_n(u_i)| \leq K(v_i - u_i) \), and

\[
|x_n(\alpha_j(v_i)) - x_n(\alpha_j(u_i))| \leq K \max \{\alpha_j(t) : t \in J\} (v_i - u_i),
\]

we have

\[
\sum_{i \in I_1} |x_n(v_i) - x_n(u_i)| \leq K \sum_{i \in I_1} (v_i - u_i) = K m(\mathcal{M}),
\]

and

\[
\sum_{i \in I_1} |x_n(\alpha_j(v_i)) - x_n(\alpha_j(u_i))| \leq K \max \{\alpha_j(t) : t \in J\} \sum_{i \in I_1} (v_i - u_i) = K \max \{\alpha_j(t) : t \in J\} m(\mathcal{M}).
\]

Consequently (cf. (6) and (7)),

\[
|\Phi| \int_{\mathcal{M}} \left( g_{1n}(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_2 n(x_n(\alpha_2(t))) \right) \, dt \\
\leq \sum_{i \in I_1} \left( \int_{x_n(v_i)}^{x_n(u_i)} (g_{1n}(s) + g_2(s)) \, ds + H_1(u_i, v_i) + H_2(u_i, v_i) \right) \\
\leq \int_{S_0} (g_1(s) + g_2(s)) \, ds + nl_{1n} \min \left\{ \frac{2}{n}, m(S_0) \right\} + K \int_{\mathcal{M}_1 \cap [0, \tau_1]} h_1(\varphi(\alpha_1(t))) \, dt \\
+ \frac{1}{\Delta_1} \int_{S_1} h_1(s) \, ds + K \int_{\mathcal{M}_1 \cap [0, \tau_2]} h_2(\varphi(\alpha_2(t))) \, dt \\
+ \frac{1}{\Delta_2} \left( \int_{S_2} h_2(s) \, ds + nl_{2n} \min \left\{ \frac{2}{n}, m(S_2) \right\} \right),
\]

where \( m(S_0) \leq K m(\mathcal{M}) < K \delta \leq \delta_0 \) and

\[ m(S_j) \leq K \max \{\alpha_j(t) : t \in J\} m(\mathcal{M}) < \delta_0 \max \{\alpha_j(t) : t \in J\}, \quad j = 1, 2. \]

Then (see (25) and (26))

\[
\int_{S_0} (g_1(s) + g_2(s)) \, ds < \frac{\varepsilon |\Phi|}{7}, \quad nl_{1n} \min \left\{ \frac{2}{n}, m(S_0) \right\} < \frac{\varepsilon |\Phi|}{7},
\]

\[
K \int_{\mathcal{M}_1 \cap [0, \tau_1]} h_1(\varphi(\alpha_1(t))) \, dt < \frac{\varepsilon |\Phi|}{7}, \quad \frac{1}{\Delta_1} \int_{S_1} h_1(s) \, ds < \frac{\varepsilon |\Phi|}{7},
\]
\[
K \int_{M_1 \cap [0, \tau_2]} h_2(\varphi(\alpha_2(t))) \, dt < \frac{\varepsilon|\Phi|}{7}, \quad \frac{1}{\Delta_2} \int_{S_2} h_2(s) \, ds < \frac{\varepsilon|\Phi|}{7},
\]
and
\[
\frac{1}{\Delta_2} nl_{2n} \min \left\{ \frac{2}{n}, m(S_2) \right\} < \frac{\varepsilon|\Phi|}{7}.
\]
Therefore
\[
\int_M \left( g_1(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_2n(x_n(\alpha_2(t))) \right) \, dt < \varepsilon, \quad n \geq n_1.
\]
By Lemma 3 with \( w_n(t) = f_{1n}(t, x_n(t), x_n(\alpha_1(t))) + f_{2n}(t, x_n(t), x_n(\alpha_2(t))), \) \( p_n(t) = g_1(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_2n(x_n(\alpha_2(t))) \) for \( n \geq n_1 \) and \([\alpha, \beta] = [0, t_*],\) (23) and (24) are true. In addition, since \( t_\ast \in (0, \gamma), \) it follows that
\[
\lim_{n \to \infty} \int_{0}^{t} \left( f_{1n}(s, x_n(s), x_n(\alpha_1(s))) + f_{2n}(s, x_n(s), x_n(\alpha_2(s))) \right) \, ds
\]
\[
= \int_{0}^{t} \left( f_1(s, x(s), x(\alpha_1(s))) + f_2(s, x(s), x(\alpha_2(s))) \right) \, ds
\]
for \( t \in [0, \gamma). \) The sequence \( \{x'_n(0)\} \) is bounded, and so we can assume that this sequence is convergent, say \( \lim_{n \to \infty} x'_n(0) = C. \) Then taking the limit as \( n \to \infty \) in the equalities
\[
x_n(t) = x_n(0) + x'_n(0)t
\]
\[
+ \mu \int_{0}^{t} \int_{0}^{s} q(v) \left( f_{1n}(v, x_n(v), x_n(\alpha_1(v))) + f_{2n}(v, x_n(v), x_n(\alpha_2(v))) \right) \, dv \, ds,
\]
we get
\[
(27) \quad x(t) = a + Ct + \mu \int_{0}^{t} \int_{0}^{s} q(v) \left( f_{1}(v, x(v), x(\alpha_1(v))) + f_{2}(v, x(v), x(\alpha_2(v))) \right) \, dv \, ds
\]
for \( t \in [0, \gamma). \) Hence \( x \in C^1([0, \gamma]). \)

We are going to prove that \( \gamma = T. \) Assume \( \gamma < T. \) From \( x_n(\gamma) - x_n((T + \gamma)/2) = x'_n(c_n)(\gamma - T)/2, \) where \( c_n \in (\gamma, (T + \gamma)/2), \) and \( \lim_{n \to \infty} x_n(\gamma) = \lim_{n \to \infty} x_n((T + \gamma)/2) = 0 \) we see that \( \lim_{n \to \infty} x'_n(c_n) = 0. \) Then integrating the inequalities (for \( n \geq n_1 \))
\[
(28) \quad 2x''_n(t)x'_n(t) \geq 2\mu T Q \left( g_1(x_n(t)) + h_1(x_n(\alpha_1(t))) + g_2(x_n(t)) + h_2n(x_n(\alpha_2(t))) \right)x'_n(t)
\]
from 0 to \( c_n \) and applying the same procedure as in the proof of Lemma 1 (now with \( x_n \) and \( c_n \) instead of \( x \) and \( T, \) respectively), we have
\[
(x'_n(c_n))^2 - (x'_n(0))^2 \geq \mu_T \left( 2Ax'_n(0) - B - 2l_{1n} - \frac{2l_{2n}}{\Delta_2} \right),
\]
where \( A \) is given by (18) and \( B \) by (19). Letting \( n \to \infty \) yields
\[
(29) \quad -C^2 \geq \mu_T (2AC - B)
\]
Suppose \( T \) is nondecreasing and
\[
a - \frac{1}{n} - x_n(c_n) = x_n(0) - x_n(c_n) = -x'_n(\eta_n)c_n,
\]
where \( \eta_n \in (0, c_n) \). Hence
\[
x'_n(0) \leq x'_n(\eta_n) = \frac{n(x_n(c_n) - a) + 1}{nc_n}
\]
and then from \( \lim_{n \to \infty} x_n(c_n) = 0 \) and \( 1/c_n > 2/(T + \gamma) \) it follows that
\[
C < -\frac{2a}{T + \gamma} < -\frac{a}{T}.
\]
Consequently, \( a/T < -C \leq \mu TA + \sqrt{\mu T A^2 + \mu T B} \) and then \( \mu T > a^2/[(2aA + BT)T] \), contrary to the definition (17) of \( \mu T \). Hence \( \gamma = T \).

Since \( \{x'_n(T)\}_{n \geq n_1} \) is bounded, we can assume that the sequence \( \{x'_n(T)\}_{n \geq n_1} \) is convergent, say \( \lim_{n \to \infty} x'_n(T) = D (\leq 0) \). Repeated integration of (28) now from 0 to \( T \) gives
\[
(x'_n(T))^2 - (x'_n(0))^2 \geq \mu T \left( 2Ax'_n(0) - B - \frac{2l_1n}{\Delta_2} \right),
\]
and letting \( n \to \infty \) we obtain \( 0 \geq D^2 - C^2 \geq \mu T(2AC - B) \). Then
\[
D^2 - C^2 \geq \frac{(2AC - B)a^2}{(2aA + BT)T}
\]
and
\[
(30) \quad C^2 + \frac{2a^2A}{(2aA + BT)T} C - \frac{a^2B}{(2aA + BT)T} - D^2 \leq 0.
\]
Now from the inequalities \( x_n(t) \leq a - (1/n) + [(3 - 2na)/(2nT)]t \) for \( t \in J \) and \( n \geq n_1 \) which follows from \( x_n(a) = a - (1/n) \), \( x_n(T) = 1/(2n) \) and \( x'' \geq 0 \) on \( (0, T) \), we have
\[
(31) \quad x(t) \leq a - \frac{at}{T} \quad \text{for} \quad t \in J.
\]
Suppose \( D = 0 \). Since \( x'_n(0) \leq (3 - 2na)/(2nT) \) for \( n \geq n_1 \), we see that \( C = \lim_{n \to \infty} x'_n(0) \leq -a/T \). On the other hand (30) implies \( C \geq -a/T \). Therefore \( C = -a/T \). Now (27) and (31) give \( x(t) = a - (at/T) \) for \( t \in J \) and
\[
\mu \int_0^t \int_0^s q(v) \left( f_1(v, x(v), x(\alpha_1(v))) + f_2(v, x(v), x(\alpha_2(v))) \right) dv \, ds = 0
\]
for \( t \in J \) and \( \mu \in [0, \mu_T] \). We see that in this case \( x(t) = a - (at/T) \) is a solution of BVP (1), (2) in the set \( \mathcal{E} \). Let \( D < 0 \). Then there exists \( n_3 \geq n_1 \) such that \( x'_n(T) \leq D/2 \) for \( n \geq n_3 \). To prove that the sequence
\[
\{f_1n(t, x_n(t), x_n(\alpha_1(t))) + f_2n(t, x_n(t), x_n(\alpha_2(t)))\}_{n \geq n_3} \subset L_1(J)
\]
has uniformly continuous integrals on \( J \) we can use the same procedure as above, now with \( D/2 \) instead of \( \Phi \). Then \( f_1(t, x(t), x(\alpha_1(t))) + f_2(t, x(t), x(\alpha_2(t))) \in L_1(J) \), by Lemma 3, and (27) shows that \( x \in C^1(J) \). We know that \( x \) is decreasing on \( J \), and so there exists a unique \( t_0 \in (0, T) \) such that \( x(t_0) = b \) and, if \( x(\alpha_2(T)) < c \), then there exists a unique \( t_1 \in (0, T) \) such that \( x(\alpha_2(t_1)) = c \). From (27) we see that \( x \) satisfies (1) for \( t \in (0, T) \) with the exception at most two points \( t_0 \) and \( t_1 \). Hence \( x \) is a solution of BVP (1), (2) in the set \( \mathcal{E} \).

From Theorem 1 it follows immediately the following result for the solvability of BVP (3), (4) in the set \( \mathcal{E} \).

**Corollary 1.** Suppose that assumptions \( (H_1) \) and

\[
(H_6) \ f_* : J \times D_1 \to [0, \infty) \text{ is continuous with } D_1 = (0, b) \cup (b, a), \ f_*(t, x) \leq g_*(x),
\]

\( (t, x) \in J \times D_1, \) where \( g_* \geq 0 \) is continuous on \( D_1 \), \( \int_0^a g_*(s) \, ds < \infty \) and

\[
\lim_{v \to 0} v g_*(b + v) = 0
\]

are satisfied. If \( 0 \leq \mu \leq a^2/[2QT \int_0^a g_*(s) \, ds] \) then BVP (3), (4) has a solution in the set \( \mathcal{E} \).

**Remark 1.** Assumption \( (H_5) \) can be replaced by the following slightly weaker assumption: There exist sequences \( \{c_n\}, \{c_n^*\}, \{d_n\}, \{d_n^*\} \) of positive numbers, \( \lim_{n \to \infty} c_n = \lim_{n \to \infty} c_n^* = \lim_{n \to \infty} d_n = \lim_{n \to \infty} d_n^* = 0 \) such that

\[
\lim_{n \to \infty} (c_n g_1(b - c_n) + d_n g_1(b + d_n)) = 0, \quad \lim_{n \to \infty} (c_n^* g_2(c - c_n^*) + d_n^* g_2(c + d_n^*)) = 0.
\]

**Remark 2.** It follows from our considerations that the assertion of Theorem 1 is true also in the cases that \( f_1 \) and \( f_2 \) have a finite number of singularities on \( (0, a) \) in the phase variables \( x \) and \( y \), respectively, supposing that singularities are of the type as \( x = b \) and \( y = c \) in assumptions \( (H_2), (H_3) \) and \( (H_5) \).

**Example 1.** Consider the BVP

\[
x''(t) = \mu(1 + |\sin(t^{-1}(T - t)^{-1})|) \left( \frac{K}{|a - 2x(t)|^\alpha} + \frac{L}{(x(t))^\beta(a - x(t))^{\gamma}} + \frac{M}{(a - x(t - T/4))^{\delta}} + \frac{N}{|a - 4x(4t - T/2)|^{\varepsilon}} \right),
\]

(32)

\[
x(t) = a \left( 1 + \frac{t}{T} \right) \quad \text{for } t \in [-T/2, 0], \quad x(T) = 0,
\]

(33)

where \( K, L, M, N \) are nonnegative constants, \( K + L + M + N > 0 \), \( \alpha, \delta, \varepsilon \in (-\infty, 1) \) and \( \beta, \gamma \in (0, 1) \). Assumptions \((j), (jj)\) and \( (H_1)-(H_5) \) are satisfied with \( \alpha_1(t) = t - (T/4), \alpha_2(t) = (4t/3) - (T/2), \Delta_1 = 1, \Delta_2 = 4/3, \tau_1 = T/4, \tau_2 = 3T/8, \varphi(t) = a(1 + (t/T)), b = a/2, c = a/4, Q = 2, g_1(x) = K/|a - 2x|^\alpha + L/[x^\beta(a - x)^\gamma], h_1(y) = \)
$M/(a - y)\delta$, $g_2(x) = 0$ and $h_2(y) = N/|a - 4y|\varepsilon$. Consequently, by Theorem 1, BVP (32),(33) has a solution in the set $\mathcal{E}$ if $0 \leq \mu \leq \mu_T$, where $\mu_T$ is defined by (17) with

$$A = 2T \left( \frac{M}{4^{1-\delta}(1 - \delta)a^\delta} + \frac{3N(3^{1-\varepsilon} - 1)}{16(1 - \varepsilon)a^\varepsilon} \right),$$

and

$$B = 4 \left( \frac{Ka^{1-\alpha}}{1 - \alpha} + La^{1-\beta-\gamma}B(1 - \beta, 1 - \gamma) + \frac{Ma^{1-\delta}}{1 - \delta} + \frac{3Na^{1-\varepsilon}(1 + 3^{1-\varepsilon})}{16(1 - \varepsilon)} \right),$$

where $B(\cdot, \cdot)$ denotes the beta function.

REFERENCES


