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QUALITATIVE ANALYSIS OF NONLINEAR RETARDED DIFFERENTIAL EQUATIONS OF SECOND ORDER

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ABSTRACT. By this research paper, we construct Lyapunov–Krasovskii functional and Lyapunov function, respectively, for non-linear functional differential equations (FDEs) of second order, in other words, for delay differential equations (DDEs) of second order, with constant retardations. By that auxliary functional and function we are able to establish five new results on the global stability and eventually uniform boundedness of solutions, square integrability of the first derivative of solutions, the existence of the periodic solutions and the existence and the uniqueness of the stationary oscillation. Here, the arguments of discussion are based only upon the second method of Lyapunov–Krasovskii functional and Lyapunov function approaches by applying some well-known theorems for the qualitative analysis of solutions of FDEs. The results introduced improve and extend the main results of the papers ([18, 53]) obtained for FDEs before. In particular case of FDEs, illustrative examples and their graphs are given to highlight the applicability of the results introduced.

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Key Words and Phrases: FDEs; second order; multiple constant retardations; global stability; eventually uniform boundedness; existence of periodic solutions; square integrability; stationary oscillation.

1. INTRODUCTION

Before and during recent years, DDEs of second order have become very important tools in mathematical modeling to real life problems and real world phenomena (Burton [3], Hale and Verduyn Lunel [5], Krasovskii [9], Smith [20] and Yoshizawa ([47, 48]). Then, the study of the qualitative properties of that kind of FDEs has gained importance during last years. Therefore, the problems of the qualitative analysis of solutions of retarded FDEs are not only as very important and interesting background in applications, but also of considerable significance in theory of ODEs

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and FDEs. Furthermore, the scope of retarded FDEs is very general, for example, the scope of retarded FDEs contains ODEs, differential-difference equations, integro-differential equations and some others. In consequence, the qualitative theory of retarded FDEs creates an important branch of nonlinear analysis for real life problems and real world phenomena. This is why so many authors focus highly of the problems on the various qualitative analysis of solutions and a series of excellent results has been achieved on the mentioned topics for various ODEs and FDEs of second order ([1]-[58]). As a specific information, retarded FDEs of second order arise in several engineering and scientific disciplines as the mathematical modelling of systems and process such as physics, biophysics, mechanics, biology, economics, engineering, atomic energy, chemistry, control theory, information theory, medicine, population dynamics and so on (Burton [3], Hale and Verduyn Lunel [5], Krasovskii [9], Smith [20] and Yoshizawa ([47, 48]). In fact, over the last forty years, there have been obtained many interesting and important results on the various and different qualitative analysis of solutions of retarded FDEs of second order. Giving the details of so more works done on the qualitative analysis of solutions of retarded FDEs of second order is a very difficult task. However, here, we would only like to present some works that can be found in the literature, and these are related to the some problems considered in this paper.

By auxiliary functions, the existence of periodic solutions of the following non-linear DDEs

$$\frac{d^2x}{dt^2} + \phi(t, \frac{dx}{dt}) + f(x(t-\tau)) = p(t) \tag{1.1}$$

and

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + g(x(t-\tau)) = p(t) \tag{1.2}$$

are investigated by Yoshizawa [47] and Zhao at al. [53], respectively. Yoshizawa [47] and Zhao at al. [53] obtained very interesting results on the existence of the period solutions of these equations.

Tunç and Yazgan [44] discussed the same topic for the below FDE of second order

$$\frac{d^2x}{dt^2} + [f(x, \frac{dx}{dt}) + g(x, \frac{dx}{dt})\frac{dx}{dt}]\frac{dx}{dt} + h(x) + \sum_{i=1}^{n} g_i(x(t-\tau_i)) = p(t).$$

By an auxiliary functional, the authors derived a new result on the existence of periodic solutions of this DDE. By this work, they extended and improved some former results in the literature.

By the same way, the following non-linear DDE of second order

$$\frac{d}{dt}[a(t)\frac{dx}{dt}] + \phi(t, \frac{dx}{dt}) + h(t, \frac{d}{dt}x(t-\tau)) + g(x) + f(x(t-\tau)) = e(t, x, \frac{dx}{dt})$$

has been considered by Tunç and Erdur [38]. In [38], the authors obtained some new qualitative results by the direct method of Lyapunov–Krasovskii functional approach. Finally, Peng [18] considered the following DDE of second order

$$\frac{d^2x}{dt^2} + f(x, \frac{dx}{dt}) + g(x, \frac{dx}{dt})\psi(x(t-\tau)) = p(t). \tag{1.3}$$

In [18], some new qualitative results are derived on the mentioned properties of solutions of DDE (1.3), except square integrability, by means of auxiliary functional and function approaches. The results from the existing literature are generalized by Peng [18]. However, to best of the information of the authors of this paper, the results of Peng [18] are not correct for DDE (1.3), but the results of [18] are correct for particular cases of DDE (1.3).

The aim of this paper is to discuss qualitative properties of solutions, which are mentioned in the abstract of this paper, for the below nonlinear DDEs with two retarded arguments,

$$\begin{aligned} \frac{d^2x}{dt^2} + f(t, x, \frac{dx}{dt}) + g(x)\psi(x(t-\tau)) + h(x)\phi(x(t-\sigma)) \\ + q(\frac{dx}{dt}) + r(x) = p(t, x, \frac{dx}{dt}), \end{aligned} \tag{1.4}$$

where $x \in \Re$, $\Re = (-\infty, \infty), t \in \Re^+, \Re^+ = [0, \infty)$, τ and σ are positive constants such that $\tau \neq \sigma$, $q, r \in C(\Re, \Re)$, $\psi, \phi \in C^1(\Re, \Re)$, $g, h \in C(\Re, \Re)$ and $f, p \in C(\Re^+ \times \Re^2, \Re)$ are functions such that $q(0) = 0$, $r(0) = 0$, $g(0)\psi(0) = 0$, $h(0)\phi(0) = 0$, $f(t, 0, 0) = 0$, $t - \tau \geq 0$ and $t - \sigma \geq 0$. In this case, the homogenous equation related to DDE (1.4) includes the zero solution. It should be mentioned that the continuity of the functions q, r, ψ, ϕ, f, g, h and p guarantees the existence of the solution of DDE (1.4). Besides, it is supposed as a basic hypothesis that the functions q, r, ψ, ϕ, g, f, h and p satisfy the Lipschitz condition with respect to the unknown functions x and its derivative x'. Hence, the uniqueness of solutions of DDE (1.4) is guaranteed.

Motivated by previous works, the aim of this paper is to extend and improve the results in [18, 47, 53] by the direct methods of Lyapunov–Krasovskii functional and Lyapunov function approaches applying to the equations of the form DDE (1.4). It is obvious that DDE (1.4) is a more general equation than DDE (1.3) when $g(x, \frac{dx}{dt}) = g(x)$ in DDE (1.3). To the authors' knowledge, the mentioned qualitative analysis of solutions for the equations of the form DDE (1.4) is investigated in the literature for the first time. In addition, there seems to be very little works available in the literature concerning the existence of periodic solutions of retarded FDEs equations of second order by the direct method of Lyapunov or Lyapunov–Krasovskii functional approach. The possible reason for this fact is the difficulty of the study of the existence of the periodic solutions for that kind of equations by these methods due to finding suitable auxiliary function(s) or functional(s). This fact is the next the main motivation for this paper. Moreover, the applicability of the hypotheses will be derived on the qualitative analysis of the solutions here by possible auxiliary function(s) or functional(s) could be very suitable rather than that established by fixed point method, formula of the variations of parameters and so on. We would not like to give any details of information here. In addition, to the best of our information the results of Peng [18] are not true for the general cases considered in [18]. Because, the derivatives of Lyapunov functional and Lyapunov function given in [18] are not correct. The results of Peng [18] can only hold true when $g(x, \frac{dx}{dt}) = g(x)$ for the DDEs considered in [18]. Our results also revise and correct the results of [18]. Here, we would not like to give a detailed discussion about these facts. Finally, the results of [18] are proved without giving any example to show and illustrate the applicability of the results of [18]. Here, we introduce two examples with their graphs. These are the main contributions of this article to the relevant topics and literature.

Next, Let $y(t) = x'(t)$. Hence, DDE (1.4) can be transformed into the following differential system

$$
\begin{aligned}
x' &= y, \\
y' &= -f(t,x,y) - g(x)\psi(x) - h(x)\phi(x) - q(y) - r(x) \\
&\quad + g(x)\int_{-\tau}^{0} \psi_x'(x(t+\eta))y(t+\eta)d\eta \\
&\quad + h(x)\int_{-\sigma}^{0} \phi_x'(x(t+\eta))y(t+\eta)d\eta + p(t,x,y).
\end{aligned} \tag{1.5}
$$

2. **BASIC INFORMATION**

Consider the DDE

$$
\frac{dx}{dt} = F(t, x_t),\ x_t = x(t+\theta),\ -h \le \theta \le 0,\ t \ge 0, \tag{2.1}
$$

where $F : \Re \times C \to \Re^n$, $C = C([-h, 0], \Re^n)$, h is a positive constant, the function F is continuous, ω-periodic such that initial value problem with respect to DDE (2.1) has a unique solution, and h can be either larger than ω or equal to or smaller than ω.

Definition 2.1. (Burton [3]). Solutions of DDE (2.1) are uniform bounded at $t = 0$ if for each positive constant B_1 there exists a positive constant B_2 such that $[\phi \in C, \|\phi\| < B_1, t \ge 0]$ imply that $|x(t, 0, \varphi)| < B_2$.

Definition 2.2. (Burton [3]). Solutions of DDE (2.1) are uniform ultimate bounded for constant bound B at $t = 0$ if for each constant $B_3 > 0$ there exists a constant $K > 0$ such that $[\phi \in C, \|\phi\| < B_3, t \ge K]$ imply that $|x(t, 0, \varphi)| < B$.

Theorem 2.3. *(Zhao et al.* [53]*). If the solutions of DDE* (2.1) *are ultimately bounded by the bound B, then DDE* (2.1) *has an ω-periodic solution and is bounded by B. If equation* (2.1) *is autonomous, then DDE* (2.1) *has an equilibrium solution and is bounded by B.*

Let $x_t = x(t)$ in DDE (2.1).

Theorem 2.4. *(Yoshizawa ([47, 48]). If there exists a positive constant B such that all solutions $\phi(t, x)$ of DDE* (2.1) *satisfy $\|\varphi(t, x)\| \leq B$, all $t \geq T_0$, where T_0 may depend on x and B, then DDE* (2.1) *has an ω-periodic solution.*

Theorem 2.5. *(Burton* [3]*). Let $V(t, x_t)$ be a differentiable scalar functional defined when $x : [\alpha, t] \to \Re^n$ is continuous and bounded by some $D \leq \infty$. If*

$$V(t, 0) = 0,\ W_1(|x(t)|) \leq V(t, x_t), \quad \textit{(where } W_1(r) \textit{ is a wedge),}$$

and

$$\dot{V}(t, x_t) \leq 0$$

then the zero solution of DDE (2.1) *is stable.*

3. QUALITATIVE ANALYSIS

We first prove a theorem on the global stability of the solutions by the Lyapunov–Krasovskii functional approach (Krasovskii [9]).

A. Hypotheses

(H1) There are positive constants r_0 and q_0 such that

$$r(0) = 0,\ \frac{r(x)}{x} \geq r_0 \quad \text{for all } x \neq 0,$$

$$q(0) = 0,\ \frac{q(y)}{y} \geq q_0 \quad \text{for all } y \neq 0.$$

(H2) $g(0)\psi(0) = 0$, $h(0)\phi(0) = 0$,

$$g(x)\psi(x)x > 0 \ \text{ and } \ h(x)\phi(x)x > 0 \ \text{ for all } \ x \neq 0.$$

(H3) There are positive constants C and D such that

$$|g(x)\psi'(u)| \leq C,\ |h(x)\phi'(u)| \leq D \ \text{ for all } \ x, u \in \Re$$

and

$$(C\tau + D\sigma)y^2 < f(t, x, y)y \ \text{ for all } \ t \in \Re^+, y \neq 0 \text{ as } \ x \in \Re.$$

(H4)

$$\int_0^x g(\xi)\psi(\xi)d\xi \to +\infty \quad \text{as} \quad |x| \to +\infty,$$

$$\int_0^x h(\xi)\phi(\xi)d\xi \to +\infty \quad \text{as} \quad |x| \to +\infty,$$

$$\int_0^x r(\xi)d\xi \to +\infty \quad \text{as} \quad |x| \to +\infty \text{ and } \int_0^y q(\zeta)d\zeta \to +\infty \quad \text{as} \quad |y| \to +\infty.$$

Let $p(t, x, y) = 0$ in DDE (1.4).

We can now state the first new result on the qualitative analysis of solutions for DDE (1.4) is the following theorem, Theorem 3.1.

Theorem 3.1. *Suppose that hypotheses (H1)–(H4) hold. Then, the zero solution of DDE* (1.4) *is globally stable.*

Proof. To prove this theorem, we benefit from the direct method of Lyapunov. Hence, we present a Lyapunov–Krasovskii functional $V = V(x, y)$ by

$$\begin{aligned} V(x,y) &= \frac{1}{2}y^2 + \int_0^x r(\xi)d\xi + \int_0^y q(\zeta)d\zeta + \int_0^x g(\xi)\psi(\xi)d\xi \\ &+ \int_0^x h(\xi)\phi(\xi)d\xi + \frac{1}{2}C\int_{-\tau}^0\int_{\tau+\eta}^t y^2(\theta)d\theta ds \\ &+\frac{1}{2}D\int_{-\sigma}^0\int_{\sigma+\eta}^t y^2(\theta)d\theta ds. \end{aligned} \tag{3.1}$$

Benefit from the hypotheses of Theorem 3.1, we observe

$$V(x, y) \geq 0,\ V(x, y) = 0 \text{ if and only if } x = y = 0.$$

Next, we have

$$V(x, y) \to \infty \quad \text{as} \quad x^2 + y^2 \to \infty.$$

By the derivative of the auxiliary functional V in (3.1) along solutions of system (1.5), we have

$$\begin{aligned} \frac{d}{dt}V &= -f(t,x,y)y + yg(x)\int_{-\tau}^0 \psi_x'(x(t+\eta))y(t+\eta)d\eta \\ &+yh(x)\int_{-\sigma}^0 \phi_x'(x(t+\eta))y(t+\eta)d\eta \\ &+\frac{1}{2}C\int_{-\tau}^0 [y^2(t) - y^2(t+\eta)]d\eta + \frac{1}{2}D\int_{-\sigma}^0 [y^2(t) - y^2(t+\eta)]d\eta. \end{aligned}$$

By means of the hypotheses of Theorem 3.1 and the inequality $2\,|ab| \le a^2+b^2$, it follows that

$$\begin{aligned}
\frac{d}{dt}V \;&\le\; -f(t,x,y)y+\int_{-\tau}^{0}|g(x(t))\psi_x'(x(t+\eta))|\,|y(t)|\,|y(t+\eta)|\,d\eta\\
&\quad+\int_{-\sigma}^{0}|h(x(t))\phi_x'(x(t+\eta))|\,|y(t)|\,|y(t+\eta)|\,d\eta\\
&\quad+\frac{1}{2}C\int_{-\tau}^{0}[y^2(t)-y^2(t+\eta)]d\eta+\frac{1}{2}D\int_{-\sigma}^{0}[y^2(t)-y^2(t+\eta)]d\eta\\
&\le\; -f(t,x,y)y+C\int_{-\tau}^{0}|y(t)|\,|y(t+\eta)|\,d\eta+D\int_{-\sigma}^{0}|y(t)|\,|y(t+\eta)|\,d\eta\\
&\quad+\frac{1}{2}C\int_{-\tau}^{0}[y^2(t)-y^2(t+\eta)]d\eta+\frac{1}{2}D\int_{-\sigma}^{0}[y^2(t)-y^2(t+\eta)]d\eta\\
&\le\; -f(t,x,y)y+\frac{1}{2}C\int_{-\tau}^{0}[y^2(t)+y^2(t+\eta)]d\eta\\
&\quad+\frac{1}{2}D\int_{-\sigma}^{0}[y^2(t)+y^2(t+\eta)]d\eta+\frac{1}{2}C\int_{-\tau}^{0}[y^2(t)-y^2(t+\eta)]d\eta\\
&\quad+\frac{1}{2}D\int_{-\sigma}^{0}[y^2(t)-y^2(t+\eta)]d\eta\\
&=\; -f(t,x,y)y+(C\tau+D\sigma)y^2\le 0. \qquad (3.2)
\end{aligned}$$

That is, we have

$$\frac{d}{dt}V(x,y)\le 0.$$

Next, in view of system (1.5) and the inequality (3.2), we can conclude that

$$\frac{d}{dt}V(x,y)=0 \quad \text{if and only if} \quad x=y=0.$$

Hence, the zero solution of DDE (1.4) is globally stable. This estimate completes the proof of Theorem 3.1. □

Example 3.2. As a particular case of DDE (1.4), let us consider the following nonlinear retarded differential equation with constant retardation, $\tau = 1/2$;

$$\begin{aligned}
&x''+\exp(-x^2)x'+24x'+\frac{x'}{(x')^2+1}\\
&+[22+\exp(-x^2)]x(t-\frac{1}{2})+x'+(x')^3+x+x^3=0. \qquad (3.3)
\end{aligned}$$

Let $y = x'$. Then DDE (3.3) can be written as the following differential system

$$
\begin{aligned}
x' &= y,\\
y' &= -[\exp(-x^2) + 24 + \frac{1}{y^2+1}]y - [22 + \exp(-x^2)]x\\
&\quad +[22 + \exp(-x^2)]\int_{-1/2}^{0} y(t+\eta)d\eta\\
&\quad -y - y^3 - x - x^3.
\end{aligned}
\tag{3.4}
$$

When we compare system (3.4) with system (1.5), it can be seen the following relations, respectively:

$$f(t,x,y) = [24 + \exp(-x^2) + \frac{1}{y^2+1}]y, \quad f(0,0) = 0,$$

$$\frac{f(t,x,y)}{y} = 24 + \exp(-x^2) + \frac{1}{y^2+1} \geq 24 > 24 \times \frac{1}{2} = 12, y \neq 0, C = 24, \tau = \frac{1}{2}.$$

That is, we can have the following estimate:

$$C\tau y^2 < f(t,x,y)y \text{ for all } t \in \Re^+, y \neq 0 \text{ as } x \in \Re.$$

Next, we observe

$$
\begin{gathered}
g(x) = 22 + \exp(-x^2),\\
\psi(x) = x,\ \psi'(x) = 1,\\
g(0)\psi(0) = 0,\\
g(x)\psi(x)x = [22 + \exp(-x^2)]x^2 > 0 \text{ for all } x \neq 0,\\
|g(x)\psi'(x)| = 22 + \exp(-x^2) \leq 23 = C,
\end{gathered}
$$

$$
\begin{aligned}
\int_0^x g(\xi)\psi(\xi)d\xi &= \int_0^x [22\xi + \xi\exp(-\xi^2)]d\xi\\
&= 11x^2 - \frac{1}{2}\exp(-x^2) \to +\infty
\end{aligned}
$$

as $|x| \to \infty$. This fact can be seen from the inequality such as

$$-\frac{1}{2} + 11x^2 \leq 11x^2 - \frac{1}{2}\exp(-x^2) \leq \frac{1}{2}\exp(-x^2) + \frac{23}{2}x^2$$

for all $x \in \Re$,

$$q(y) = y + y^3, q(0) = 0, \frac{q(y)}{y} = 1 + y^2 \geq 1 = q_0 \text{ for all } y \neq 0,$$

$$r(x) = x + x^3, r(0) = 0, \frac{r(x)}{x} = 1 + x^2 \geq 1 = r_0 \text{ for all } x \neq 0.$$

Hence, the discussion made above shows that all the hypotheses of Theorem 3.1, (H1)–(H4) can be verifiable. Then the zero solution of DDE (3.3) is globally stable.

This case, that is, the globally stability of solutions of DDE (3.3), hence the same behavior for the equivalent system (3.4) , can be observed from Figure 1 and Figure 2.

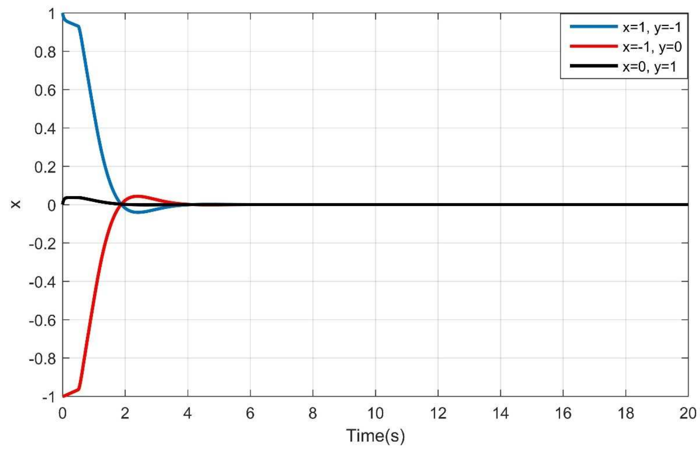

FIGURE 1

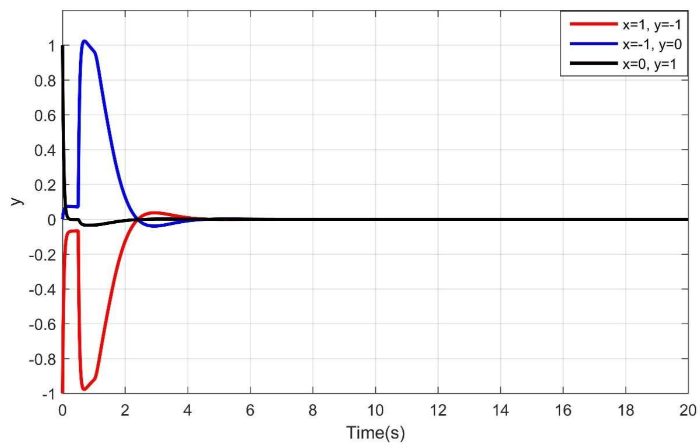

FIGURE 2

We now introduce the rest of our hypotheses.

B. Hypotheses

(H5) There are positive constants m, n, N and M such that

$$m \leq g(x) \leq M \quad \text{and} \quad n \leq h(x) \leq N \quad \text{for all } x \in \Re,$$

(H6) There are positive constants C_0 and D_0 such that

$$|g(x)\psi'(u)| \leq C_0 \quad \text{and} \quad |h(x)\phi'(u)| \leq D_0 \quad \text{for all} \quad x, u \in \Re,$$

(H7) There are positive constants C_1 and H such that

$$f(t,0,0) = 0, \frac{f(t,x,y)}{y} \geq C_1 \quad \text{for all } t \in \Re^+, x \in \Re \quad \text{as} \quad |y| \geq H,$$

(H8)

$$\tau C_0 \sqrt{\frac{M}{m}} + \sigma D_0 \sqrt{\frac{N}{n}} < C_1,$$

where m, n, C_0, D_0, C_1, N and M are some constants, which are given in hypotheses (H5)–(H7).

(H9) There is a continuous and bounded function $\sigma_1(t)$ and there are positive constants C_2, C_3 and K such that

$$|p(t,x,y)| \leq |\sigma_1(t)|\,|y| \quad \text{for all } t \in \Re^+ \text{ and } x, y \in \Re$$

and

$$\frac{f(t,x,y)}{y} - \tau C_0 C_2 \sqrt{\frac{M}{m}} - \sigma D_0 C_3 \sqrt{\frac{N}{n}} - |\sigma_1(t)| \geq K$$

for all $t \in \Re^+, x \in \Re$ as $|y| \geq H$.

Let $p(t,x,y) \neq 0$ in DDE (1.4).

We second prove four theorems on the qualitative analysis of the solutions by the Lyapunov-function approach (Burton [3]).

The next new result on the qualitative analysis of solutions is the following theorem.

Theorem 3.3. *If hypotheses (H1) and (H5)–(H9) hold, then every solution of DDE* (1.4) *is eventually uniform bounded.*

Proof. To prove this theorem, we benefit from the second method of Lyapunov. Hence, define the Lyapunov function $V_1 = V_1(x,y)$ by

$$V_1(x,y) = \frac{1}{2}y^2 + \int_0^x r(\xi)d\xi + \int_0^y q(\zeta)d\zeta + \int_0^x g(\xi)\psi(\xi)d\xi + \int_0^x h(\xi)\phi(\xi)d\xi. \tag{3.5}$$

In view of the hypotheses of Theorem 3.3 and the function $V_1(x,y)$, it is clear that

$$V_1(x,y) \geq 0,\ V_1(x,y) = 0 \text{ if and only if } x = y = 0.$$

Further, we have

$$V_1(x,y) \to \infty \ \text{ as } \ x^2 + y^2 \to \infty.$$

The derivative of the auxiliary Lyapunov function V_1 in (3.5) through the solutions of system (1.5) gives

$$\begin{aligned} \frac{d}{dt}V_1 &= -f(t,x,y)y + yg(x)\int_{-\tau}^0 \psi_x'(x(t+\eta))y(t+\eta)d\eta \\ &\quad + yh(x)\int_{-\tau}^0 \phi_x'(x(t+\eta))y(t+\eta)d\eta + yp(t,x,y). \end{aligned}$$

Now, by hypotheses (H5)–(H9) of Theorem 3.3 and the estimate $2|ab| \leq a^2+b^2$, it follows that

$$\begin{aligned}
\frac{d}{dt}V_1 &\leq -f(t,x,y)y + \int_{-\tau}^{0} |g(x(t))\psi_x'(x(t+\eta))|\,|y(t)|\,|y(t+\eta)|\,d\eta \\
&\quad + \int_{-\sigma}^{0} |h(x(t))\phi_x'(x(t+\eta))|\,|y(t)|\,|y(t+\eta)|\,d\eta + |y|\,|p(t,x,y)| \\
&\leq -f(t,x,y)y + C_0\int_{-\tau}^{0} |y(t)|\,|y(t+\eta)|\,d\eta \\
&\quad + D_0\int_{-\sigma}^{0} |y(t)|\,|y(t+\eta)|\,d\eta + |\sigma_1(t)|\,y^2.
\end{aligned}$$

By hypothesis (H8), that is, $\tau C_0\sqrt{\frac{M}{m}} + \sigma D_0\sqrt{\frac{N}{n}} < C_1$, it follows that there exist positive numbers $C_2 > 1$ and $C_3 > 1$ such that

$$|y(t+\eta)| \leq C_2\sqrt{\frac{M}{m}}\,|y(t)| \quad \text{and} \quad |y(t+\eta)| \leq C_3\sqrt{\frac{N}{n}}\,|y(t)| \ \text{ as } |y| \geq H.$$

Hence, by hypothesis (H9), we can obtain

$$\begin{aligned}
\frac{d}{dt}V_1 &\leq -f(t,x,y)y + C_0C_2\sqrt{\frac{M}{m}}\int_{-\tau}^{0} y^2(t)d\eta \\
&\quad + D_0C_3\sqrt{\frac{N}{n}}\int_{-\sigma}^{0} y^2(t)d\eta + |\sigma_1(t)|\,y^2 \\
&= -f(t,x,y)y + \tau C_0C_2\sqrt{\frac{M}{m}}y^2 + \sigma D_0C_3\sqrt{\frac{N}{n}}y^2 + |\sigma_1(t)|\,y^2 \\
&= -\left[\frac{f(t,x,y)}{y} - \tau C_0C_2\sqrt{\frac{M}{m}} - \sigma D_0C_3\sqrt{\frac{N}{n}} - |\sigma_1(t)|\right]y^2 \\
&\leq -Ky^2 \leq 0.
\end{aligned}$$

Then

$$\frac{d}{dt}V_1(x,y) \leq 0.$$

In view of this result, we can conclude that every solution of DDE (1.4) is eventually uniformly bounded [18]. In fact, the integration of $\frac{d}{dt}V_1(x,y) \leq 0$ from t_0 to t gives

$$V_1(x,y) \leq V_1(x(t_0),y(t_0)) \quad \text{for all } t \geq t_0.$$

Since the function V_1 is positive definite, we can say that

$$V_1(x(t_0),y(t_0)) = K_1 > 0,\ K_1 \in \Re.$$

Then

$$V_1(x,y) \leq K_1,\ t \geq t_0.$$

Next, it is easy to obtain

$$\frac{1}{2}r_0x^2 + \frac{1}{2}y^2 \leq \int_0^x \frac{r(\xi)}{\xi}\xi d\xi + \frac{1}{2}y^2 = \int_0^x r(\xi)d\xi + \frac{1}{2}y^2 \leq V_1(x,y) \leq K_1$$

by hypothesis (H1). It is now clear to see that

$$|x| \leq \sqrt{2r_0^{-1}K_1} \quad \text{and} \quad |y| \leq \sqrt{2K_1}$$

for all $t \geq t_0$. From this relations, we can conclude that every solution of DDE (1.4) is eventually uniform bounded. □

Theorem 3.4. *If hypotheses (H1) and (H5)–(H9) hold, then the first derivative of every solution of DDE* (1.4) *is square integrable, that is, $y \in L^2[0,\infty)$.*

Proof. To prove this theorem, as before we benefit from the second method of Lyapunov and hence the auxiliary Lyapunov function $V_1 = V_1(x,y)$. By hypotheses (H5)–(H9) of Theorem 3.4, we can obtain

$$\frac{d}{dt}V_1(x,y) \leq -Ky^2 \leq 0.$$

Integrating this inequality from t_0 to t, we have

$$V_1(x,y) - V_1(x(t_0), y(t_0)) \leq -K\int_{t_0}^{t} y^2(s)ds.$$

Then, we arrange the last inequality as

$$K\int_{t_0}^{t} y^2(s)ds \leq V_1(x(t_0), y(t_0)) - V_1(x,y) \leq V_1(x(t_0), y(t_0)) = K_1.$$

Thus,

$$\int_{t_0}^{t} y^2(s)ds \leq K^{-1}K_1.$$

Hence, we can conclude that

$$\int_{t_0}^{\infty} y^2(s)ds < \infty.$$

This results finishes the proof of Theorem 3.4. □

Theorem 3.5. *If hypotheses (H1), (H5)–(H9) hold and in addition $f(t,x,y)$and $p(t,x,y)$ are periodic functions of T–periodicity, that is, $f(t+T,x,y) = f(t,x,y)$ and $p(t+T,x,y) = p(t,x,y)$, then DDE* (1.4) *has some periodic solutions of T–periodicity.*

Proof. By the hypotheses of Theorem 3.5, clearly, we can reach that every solution of DDE (1.4) is bounded for all $t \in (0,\infty)$. By Massera's theorem ([19]), Theorem 2.3 and Theorem 2.4, we can conclude that DDE (1.4) has some periodic solutions of T–periodicity. This results finishes the proof of Theorem 3.5. □

Example 3.6. We now modify DDE (3.3) as following nonlinear DDE of second order with constant delay, $\tau = 1/2$:

$$\begin{aligned} x'' \quad & + \exp(-x^2)x' + 24x' + \frac{x'}{(x')^2+1} \\ & +22 + \exp(-x')^2 x(t-\frac{1}{2}) + x' + (x')^3 + x + x^3 = \sin t. \end{aligned} \tag{3.6}$$

Hence, it is obvious the existence of the following estimates:

$$m = 22 \leq g(x) = 22 + \exp(-x^2) \leq 23 = M \quad \text{for all } x \in \Re,$$

$$\psi(x) = x, \psi'(x) = 1,$$

$$|g(x)\psi'_x(u)| \leq 23 = C_0 \quad \text{for all } x, u \in \Re,$$

$$f(t,x,y) = 24y + y\exp(-x^2) + \frac{y}{y^2+1},$$

$$\frac{f(t,x,y)}{y} = 24 + \exp(-x^2) + \frac{1}{y^2+1} \geq 24 = C_1 \quad \text{for all } x \in \Re \text{ as } |y| \geq H,$$

$$p(t,x,y) = \sin t,$$

$$p(t+2\pi, x, y) = \sin(t+2\pi) = \sin t,$$

$$T = 2\pi, \ |p(t,x,y)| \leq |\sin t|, \ \sigma_1(t) = \sin t.$$

$p(t,x,y)$ is a periodic function in t with the period $T = 2\pi$.

$$\tau C_0 \sqrt{\frac{M}{m}} = \frac{1}{2} \times 24 \times \sqrt{\frac{24}{22}} \cong 12,54 < 24 = C_1,$$

$$\begin{aligned} \frac{f(t,x,y)}{y} - \tau C_0 C_2 \sqrt{\frac{M}{m}} - |\sigma_1(t)| \quad & \geq \quad 24 - \frac{1}{2} \times 24 \times C_2 \times \sqrt{\frac{24}{22}} - |\sin t| \\ & \geq \quad 23 - 12 \times C_2 \times 1,05 \\ & \geq \quad 23 - 12,6 \times C_2. \end{aligned}$$

Let $C_2 = 1,1$. Then $23 - 12,6 \times C_3 = 9,4 = K$.

$$\frac{f(t,x,y)}{y} - \tau C_0 C_2 \sqrt{\frac{M}{m}} - |\sigma_1(t)| \geq 9,4 = K.$$

Thus all hypotheses of Theorems 3.3, 3.4 and 3.5 hold. Hence, we can conclude that all solutions of DDE (3.6) are eventually uniform bounded and the first derivative of every solution of DDE (3.6) is square integrable. Moreover, DDE (3.6) has some periodic solutions of 2π–periodicity. This case, that is, the eventually uniform boundedness of solutions, square integrability of the first derivatives of solutions and the existence of periodic solutions of 2π–periodicity of DDE (3.6) can be seen form Figure 3 and Figure 4.

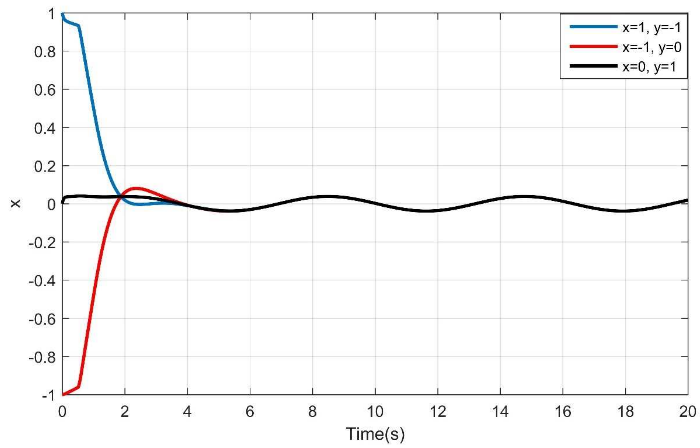

FIGURE 3

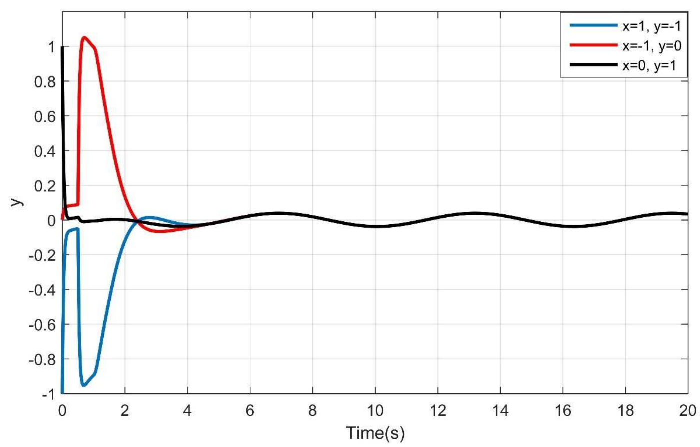

FIGURE 4

Theorem 3.7. *If hypotheses (H1), (H4)–(H8) hold and* $p(t, x, y)$ *are a periodic function of* T*–periodicity, that is,* $p(t + T, x, y) = p(t, x, y)$*, and* $x_1(t)$ *and* $x_2(t)$ *are any two solutions of DDE* (1.4) *such that* $x_1(t) - x_2(t) \to 0$ *and* $x_1'(t) - x_2'(t) \to 0$ *as* $t \to \infty$*, then there exists unique one stationary oscillation in DDE* (1.4)*.*

Proof. From Theorem 3.5, clearly, it is known that DDE (1.4) has some periodic solutions of T–periodicity. Further, from Theorem 3.7, it is known that $x_1(t)$ and $x_2(t)$ are any two solutions of DDE (1.4) such that $x_1(t) - x_2(t) \to 0$ and $x_1'(t) - x_2'(t) \to 0$ as $t \to \infty$. Hence, we can predict that there exists unique one stationary oscillation in DDE (1.4). Thus, consequently, we can conclude that there exists unique one stationary oscillation in DDE (1.4) (see [[19], Lasalle's theorem]). □

4. Discussion

A class of non-linear DDEs of second with multiple constant retardations is considered. The global stability of solutions, eventually uniform boundedness of solutions, square integrability of the first derivative of solutions, the existence of the periodic solutions, the existence and uniqueness of the stationary are investigated. The basic methods used in the qualitative analysis are the second method of Lyapunov functional and function approaches. Our results complement, correct and improve some conclusions can be found in the literature. We also give an additional result on the square inegrability of the solutions. Finally, two illustrative examples are given to verify the results introduced and their graphs are drawn by applying MATLAB-Simulink.

REFERENCES

[1] S. Ahmad, M. Rama Mohana Rao, Theory of ordinary differential equations. With applications in biology and engineering. Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.

[2] A. Ardjouni, A. Djoudi, Existence of positive periodic solutions for a second-order nonlinear neutral differential equation with variable delay. Adv. Nonlinear Anal. 2 (2013), no. 2, 151–161.

[3] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations. Academic Press, Orlando, 1985.

[4] C. Guo, Y. Xu, Existence of periodic solutions for a class of second order differential equation with deviating argument. J. Appl. Math. Comput. 28 (2008), no. 1-2, 425–433.

[5] J. K. Hale, S. M. Verduyn Lunel, Introduction to functional-differential equations. Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[6] R. Jia, J. Shao, Existence and uniqueness of periodic solutions of second-order nonlinear differential equations. J. Inequal. Appl. 2013, 2013:115, 10 pp.

[7] T. Ji, W. Dong, Existence of periodic solutions for a class of differential equations of second order. Applied and Computational Mathematics, 255–257, Math. Comput. Sci. Eng., WSEAS Press, Athens, 2008.

[8] E. Korkmaz, C. Tunç, Convergence to non-autonomous differential equations of second order. J. Egyptian Math. Soc. 23 (2015), no. 1, 27–30.

[9] N. N. Krasovskii, Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay, Stanford, Calif.: Stanford University Press 1963.

[10] Q. Li, Y. Li, On the existence of positive periodic solutions for second-order functional differential equations with multiple delays. Abstr. Appl. Anal. 2012, Art. ID 929870, 13 pp.

[11] X. Li, Z. Zhang, On the existence of positive periodic solutions of systems of second order differential equations. Math. Nachr. 284 (2011), no. 11-12, 1472–1482.

[12] B. Liu, L. Huang, Boundedness of solutions for a class of retarded Lienard equation. J. Math. Anal. Appl. 286 (2003), no. 2, 422–434.

[13] R. Ma, Y. Lu, Existence of positive periodic solutions for second-order functional differential equations. Monatsh. Math. 173 (2014), no. 1, 67–81.

[14] I. A. Malyseva, Boundedness of solutions of a Lienard differential equation. Differetia'niye Uravneniya 15 (1979), no. 8, 1420–1426.

[15] M. Murean, Boundedness of solutions for Lienard type equations. Mathematica 40 (63) (1998), no. 2, 243–257.

[16] O. G. Mustafa, C. Tunç, Asymptotically linear solutions of differential equations via Lyapunov functions. Appl. Math. Comput. 215 (8), (2009), 3076–3081.

[17] J. E. Napoles Valdes, Boundedness and global asymptotic stability of the forced Linard equation. Rev. Un. Mat. Argentina 41 (2000), no. 4, 47–59.

[18] Q. L. Peng, Qualitative analysis for a class of second-order nonlinear system with delay. Appl. Math. Mech. (English Ed.) 22 (2001), no. 7, 842–845; translated from Appl. Math. Mech. 22 (2001), no. 7, 749–752 (Chinese).

[19] R. Reissig, G. Sansone, R. Conti, Non-linear differential equations of higher order. Translated from the German. Noordhoff International Publishing, Leyden, 1974.

[20] H. Smith, An introduction to delay differential equations with applications to the life sciences. Texts in Applied Mathematics, 57. Springer, New York, 2011.

[21] J. Sugie, Y. Amano, Global asymptotic stability of non-autonomous systems of Lienard type. J. Math. Anal. Appl. 289 (2004), no. 2, 67–690.

[22] D. S. Tian, Existence of multiple periodic solutions for a second-order delayed differential equation. Acta Math. Appl. Sin. 39 (2016), no. 4, 537–546.

[23] D. S. Tian, X. W. Zeng, Existence of several periodic solutions for a class of second-order functional differential equations. Acta Math. Sci. Ser. A 30 (2010), no. 2, 525–530.

[24] C. Tunç, Some new stability and boundedness results on the solutions of the nonlinear vector differential equations of second order. Iran. J. Sci. Technol. Trans. A Sci. 30 (2006), no. 2, 213–221.

[25] C. Tunç, A new boundedness theorem for a class of second order differential equations. Arab. J. Sci. Eng. Sect. A Sci. 33 (2008), no. 1, 83–92.

[26] C. Tunç, Some stability and boundedness results to nonlinear differential equations of Lienard type with finite delay. J. Comput. Anal. Appl. 11(4), (2009), 711–727.

[27] C. Tunç, Some new stability and boundedness results of solutions of Lienard type equations with a deviating argument. Nonlinear Anal. Hybrid Syst. 4 (2010), no. 1, 85–91.

[28] C. Tunç, A note on boundedness of solutions to a class of non-autonomous differential equations of second order. Appl. Anal. Discrete Math. 4 (2010), no. 2, 361–372.

[29] C. Tunç, New stability and boundedness results of Lienard type equations with multiple deviating arguments. Izv. Nats. Akad. Nauk Armenii Mat. 45 (2010), no. 4, 47–56.

[30] C. Tunç, Boundedness results for solutions of certain nonlinear differential equations of second order. J. Indones. Math. Soc.. 16 (2010), no.2, 115–128.

[31] C. Tunç, Stability, boundedness and uniform boundedness of solutions of nonlinear delay differential equations. Discrete Contin. Dyn. Syst. 2011, Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. II, 1395–1403.

[32] C. Tunç, Stability and boundedness of solutions of non-autonomous differential equations of second order. J. Comput. Anal. Appl. 13 (2011), no. 6, 1067–1074.

[33] C. Tunç, Y. Altun, Asymptotic stability in neutral differential equations with multiple delays. J. Math. Anal. 7 (2016), no. 5, 4053.

[34] C. Tunç, On the qualitative behaviors of a functional differential equation of second order. Appl. Appl. Math. 12 (2017), no. 2, 813–2842.

[35] C. Tunç, On the properties of solutions for a system of non-linear differential equations of second order. Int. J. Math. Comput. Sci. 14 (2019), no. 2, 519–534.

[36] C. Tunç, I. Çinar, On the existence of periodic solutions to nonlinear differential equations of second order. Differ. Uravn. Protsessy Upr. 2008, no. 3, 20–25.

[37] C. Tunç, Y. Dinç, Qualitative properties of certain non-linear differential systems of second order. Journal of Taibah University for Science 11 (2017), no.2, 359–366.

[38] C. Tunç, S. Erdur, New Qualitative Results for Solutions of Functional Differential Equations of Second Order. Discrete Dyn. Nat. Soc. 2018, Art. ID 3151742, 13 pp.

[39] C. Tunç, E. Tunç, On the asymptotic behavior of solutions of certain second-order differential equations. J. Franklin Inst. 344 (2007), no. 5, 391–398.

[40] C. Tunç, O. Tunç, A note on certain qualitative properties of a second order linear differential system. Appl. Math. Inf. Sci. 9 (2015), no. 2, 953–956.

[41] C. Tunç, O. Tunç, On the boundedness and integration of non-oscillatory solutions of certain linear differential equations of second order. Journal of Advanced Research. 7 (1), (2016), 165–168.

[42] C. Tunç, O. Tunç, A note on the stability and boundedness of solutions to non-linear differential systems of second order. Journal of the Association of Arab Universities for Basic and Applied Sciences, Vol. 24, (2017), 169–175.

[43] C. Tunç, O. Tunç, Qualitative analysis for a variable delay system of differential equations of second order. Journal of Taibah University for Science. 13 (2019), no.1, 468–477.

[44] C. Tunç, R. Yazgan, Existence of periodic solutions to multidelay functional differential equations of second order. Abstr. Appl. Anal. (2013) Art. ID 968541, 5 pp.

[45] Y. Wei, Existence and uniqueness of periodic solutions for second order differential equations. J. Funct. Spaces 2014, Art. ID 246258, 5 pp.

[46] G. R. Ye, H. S. Ding, X. L. Wu, Uniform boundedness of solutions for a class of Linard equations. Electron. J. Differential Equations 2009, no. 97, 5 pp.

[47] T. Yoshizawa, Stability theory by Liapunov's second method. The Mathematical Society of Japan, Tokyo, 1966.

[48] T. Yoshizawa, Stability theory and the Existence of Periodic Solutions and Almost Periodic Solutions. Springer-Verlag, 1975.

[49] Y. Yu, B. Xiao, Boundedness of solutions for a class of Lienard equation with multiple deviating arguments. Vietnam J. Math. 37 (2009), no. 1, 35–41.

[50] Y. Yu, C. Zhao, Boundedness of solutions for a Lienard equation with multiple deviating arguments. Electron. J. Differential Equations 2009, no. 14, 5 pp.

[51] B. Zhang, Boundedness and stability of solutions of the retarded Lienard equation with negative damping. Nonlinear Anal. 20 (1993), no. 3, 303–313.

[52] L. Zhang, Q. Y. Wang, Existence of periodic solutions for a class of second-order functional differential equations with deviating arguments. Acta Math. Sci. Ser. A 29 (2009), no. 2, 253–261.

[53] J. M . Zhao, K. L. Huang, Q. S. Lu, The existence of periodic solutions for a class of functional-differential equations and their application. Appl. Math. Mech. (English Ed.) 15 (1994), no. 1, 49–59; translated from Appl. Math. Mech. 15 (1994), no. 1, 49–58 (Chinese).

[54] T. F. Zhou, Existence of periodic solutions for semi-linear second-order differential equations. Gongcheng Shuxue Xuebao 32 (2015), no. 5, 690–696.

[55] Z. F. Zhou, W. Jiang, Stability and boundedness of retarded Lienard-type equation. Chinese Quart. J. Math. 18 (2003), no. 1, 7–12.

[56] J. Zhou, L. Xiang, On the stability and boundedness of solutions for the retarded Linard-type equation. Ann. Differential Equations 15 (1999), no. 4, 460–465.

[57] L. M. Zhu, Y. X. Li, Existence of periodic solutions for second-order differential equations with multiple delays. J. Jilin Univ. Sci. 55 (2017), no. 5, 1077–1083.

[58] J. Zu, Existence and uniqueness of periodic solution for nonlinear second-order ordinary differential equations. Bound. Value Probl. 2011, Art. ID 192156, 11 pp.