Dynamic Systems and Applications 29 (2020) 111-122

ROBUST CONTROL FOR VARIABLE ORDER TIME FRACTIONAL FINANCIAL SYSTEM

A. SHABANI[1], A. H. REFAHI SHEIKHANI[1,*], AND H. AMINIKHAH[2]

[1]Department of Applied Mathematics, Faculty of Mathematical Sciences, Lahijan Branch, Islamic Azad University, Lahijan, Iran
[2]Department of Applied mathematics, School of Mathematical Sciences, University of Guilan, Rasht, Iran

ABSTRACT. In this paper, we study the chaos control for a variable-order fractional financial system using the robust control approach. For this purpose, the numerical solution of a variable order fractional financial system and the control law were calculated by using the Adams-Bashforth-Moulton method. The derivative was defined in the Atangana-Baleanu-Caputo variable-order fractional sense. To experiment the control stability efficiency, different statistical indicators were presented. Finally, simulation results established the effectiveness of the suggested robust control of the current study.

AMS (MOS) Subject Classification. 34C28, 34H10, 91G80.

1. Introduction

Fractional calculus (FC) would enable the operators in integration and differentiation to achieve fractional order. Investigating FC is regarded as a common research subject and has been a serious research topic in the past few decades [3, 4, 6, 17, 20, 24, 25, 28, 36]. Samko presented a noticeable extension of the constant order FC [27].

Conducting studies on fractional differential equations has recently turned into an active area of investigation, and many equations in interdisciplinary fields have been proven to be capable of being extensively described through fractional derivatives. References [2, 14, 22, 23] deal with stability analysis, dynamical properties, and simulation of some fractional differential equations.

Chaos is a phenomenon explaining high nonlinear behaviors. A chaotic system is sensitive to small alterations in initial conditions. Chaotic systems (for example see [19, 13] and references therein) are considered significant in such areas as control, finances, secure communication, and so forth.

Received January 26, 2020
*Corresponding Author.

The definition of a financial system could be proposed at the global, regional, or firm-specific levels and is a set of procedures applied in order to track financial activities. The financial system, on a regional scale, is the system making the lenders and borrowers capable of exchanging funds. Financial systems involve complicated and intricate models representing financial services, institutions, and markets establishing a system of inter-connections between depositors and investors in [16, 18, 35].

The author, in the present research work, proposes fractional operators that regard the order as a function of time, space, or a number of other variables. [5, 31] present the major applications of such fractional variable-order operators. Due to the fact that the exact solutions of variable-order fractional differential equations cannot found, the development of a numerical scheme that is capable of solving such equations is a crucial area of investigation. Adams-Bashforth method has conventionally been seen as a powerful, outstanding numerical method, capable of presenting the numerical solution of fractional differential equations [8, 10, 11, 12]. As is seen in [9], the authors have recently created a constant-order numerical scheme integrating the fundamental theorem of fractional calculus with the two-step Lagrange polynomial. In this approach, the numerical schemes presented in [9] are generalized in order for simulating variable-order time fractional financial systems.

The paper mainly handles robust control. This issue is then briefly discussed here. Manufacturing, mining, automobile and other hardware applications make extensive use of feedback control systems. In order to meet the ever-growing demands for higher efficiency and reliability levels, such control systems are needed to ensure more effective and accurate overall performance in the face of demanding and constantly changing operating conditions. Control engineers need new design tools and more effective control theories in order for designing control systems to meet the requirements of improved performance and robustness while controlling complex processes. A standard technique of promoting the control system performance would integrate extra sensors and actuators into the system. Such an approach would result necessarily in a multi-input multi-output (MIMO) control system. Hence any modern feedback control system design methodology must be capable of dealing with cases in which multiple actuators and sensors are involved. Moreover, a control system is robust when: (1) its sensitivity level is low, (2) it has stability over the range of parameter variations, and (3) its performance constantly meets the specifications in the presence of a set of system parameter variations [26, 30, 32, 33]. This paper is aimed at applying the variable-order fractional derivative of Atangana-Baleanu type in the chaos control for a variable-order time fractional financial system by means of robust control mode for the first time.

This paper is outlined as follows. Some required preliminaries in the sequence are presented in Section 2. Section 3 considers the numerical approach procedure. Robust control for variable-order time fractional financial system is discussed in Section 4. Section 5 deals with simulation results. Finally a brief discussion about the method and the generated results are briefly discussed in Section 6.

2. Preliminaries

In this section we give some basic tools which we need in the future. The Atangana-Baleanu fractional derivative with variable-order $q(t)$ in Liouville-Caputo sense (ABC) is defined as follows [34]:

$$ABC_0D_t^{q(t)}f(t) = \frac{B(q(t))}{1-q(t)}\int_0^t E_{q(t)}\Big[-q(t)\frac{(t-\tau)^{q(t)}}{1-q(t)}\Big]f(\tau)d\tau, \tag{2.1}$$

where $n-1 < q(t) \leq n$ and $B(q(t)) = 1-q(t)+\dfrac{q(t)}{\Gamma(q(t))}$ is a normalization function. The related integral Atangana- Baleanu can be formulated as

$$ABC_0I_t^{q(t)}f(t) = \frac{1-q(t)}{B(q(t))}q(t) + \frac{q(t)}{B(q(t))\Gamma(q(t))}\int_0^t f(\tau)(t-\tau)^{q(t)-1}d\tau, \tag{2.2}$$

in which $n-1 < q(t) \leq n$. Consider an th n^{th}-order fractional differential equation of the form

$$ABC_0D_t^{q(t)}f(t) = F(t,f(t)), \qquad f_0^{(k)} = f_0^k, \qquad k = 0,1,...,n-1. \tag{2.3}$$

This equation can be rewritten as

$$f(t) = f_0 + \frac{1-q(t)}{B(q(t))}F(t,f(t)) + \frac{q(t)}{B(q(t))\Gamma(q(t))}\int_0^t F(\tau,f(\tau))(t-\tau)^{q(t)-1}d\tau. \tag{2.4}$$

It is possible to extend the Adams method for this equation as follows:

$$\begin{aligned} f_{i+1}^p &= f_0 + \frac{1-q(t_i)}{B(q(t_i))}F(t_i,f_i) + \frac{q(t_i)}{B(q(t_i))\Gamma(q(t_i))}\sum_{j=0}^{i} b_{j,i+1}F(t_j,f_j),\\ f_{i+1} &= f_0 + \frac{1-q(t_{i+1})}{B(q(t_{i+1}))}F(t_{i+1},f_{i+1}^p) + \frac{q(t_{i+1})h^{q(t_{i+1})}}{B(q(t_{i+1}))\Gamma(q(t_{i+1})+2)}\times\\ &\Big[F(t_{i+1},f_{i+1}^p)\sum_{j=0}^{i} a_{j,i+1}F(t_j,f_j)\Big], \end{aligned} \tag{2.5}$$

where

$$b_{j,i+1} = \frac{h^{q(t_{i+1})}}{q(t_{i+1})}\Big((i-j+1)^{q(t_{i+1})} - (i-j)^{q(t_{i+1})}\Big), \qquad j = 0,1,...,i$$

and

$$a_{j,i+1} = \begin{cases} i^{q(t_{i+1})+1} - (i-q(t_{i+1}))(i+1)^{q(t_{i+1})}, & j=0,\\ (i-j+2)^{q(t_{i+1})+1} + (i-j)^{q(t_{i+1})+1} - 2(i-j+1)^{q(t_{i+1})+1}, & 1 \leq j \leq i. \end{cases}$$

3. **Numerical approach**

In this section, we develop the Adams method for the financial system

$$\begin{cases} ABC_0D_t^{q(t)}x(t) = F_1(t_i, x_i, y_i, z_i) = z(t) + (y(t) - a)x(t), \\ ABC_0D_t^{q(t)}y(t) = F_2(t_i, x_i, y_i, z_i) = 1 - by(t) - x(t)^2, \\ ABC_0D_t^{q(t)}z(t) = F_3(t_i, x_i, y_i, z_i) = -x(t) - cz(t). \end{cases} \tag{3.1}$$

Similar to the previous section, we can write

$$\begin{cases} x_{i+1}^p = x_0 + \dfrac{1-q(t_i)}{B(q(t_i))}F_1(t_i, x_i, y_i, z_i) + \dfrac{q(t_i)}{B(q(t_i))\Gamma(q(t_i))}\sum_{j=0}^{i} b_{j,i+1}F_1(t_j, x_j, y_j, z_j), \\ y_{i+1}^p = y_0 + \dfrac{1-q(t_i)}{B(q(t_i))}F_2(t_i, x_i, y_i, z_i) + \dfrac{q(t_i)}{B(q(t_i))\Gamma(q(t_i))}\sum_{j=0}^{i} b_{j,i+1}F_2(t_j, x_j, y_j, z_j), \\ z_{i+1}^p = z_0 + \dfrac{1-q(t_i)}{B(q(t_i))}F_3(t_i, x_i, y_i, z_i) + \dfrac{q(t_i)}{B(q(t_i))\Gamma(q(t_i))}\sum_{j=0}^{i} b_{j,i+1}F_3(t_j, x_j, y_j, z_j). \end{cases}$$

Also

$$\begin{cases} x_{i+1} = x_0 + \dfrac{1-q(t_{i+1})}{B(q(t_{i+1}))}F_1(t_{i+1}, x_{i+1}^p, y_{i+1}^p, z_{i+1}^p) + \dfrac{q(t_{i+1})h^{q(t_{i+1})}}{B(q(t_{i+1}))\Gamma(q(t_{i+1})+2)}\times \\ \Big[F_1(t_{i+1}, x_{i+1}^p, y_{i+1}^p, z_{i+1}^p) + \sum_{j=0}^{i} a_{j,i+1}F_1(t_j, x_j, y_j, z_j)\Big], \\ y_{i+1} = y_0 + \dfrac{1-q(t_{i+1})}{B(q(t_{i+1}))}F_2(t_{i+1}, x_{i+1}^p, y_{i+1}^p, z_{i+1}^p) + \dfrac{q(t_{i+1})h^{q(t_{i+1})}}{B(q(t_{i+1}))\Gamma(q(t_{i+1})+2)}\times \\ \Big[F_2(t_{i+1}, x_{i+1}^p, y_{i+1}^p, z_{i+1}^p) + \sum_{j=0}^{i} a_{j,i+1}F_2(t_j, x_j, y_j, z_j)\Big], \\ z_{i+1} = z_0 + \dfrac{1-q(t_{i+1})}{B(q(t_{i+1}))}F_3(t_{i+1}, x_{i+1}^p, y_{i+1}^p, z_{i+1}^p) + \dfrac{q(t_{i+1})h^{q(t_{i+1})}}{B(q(t_{i+1}))\Gamma(q(t_{i+1})+2)}\times \\ \Big[F_3(t_{i+1}, x_{i+1}^p, y_{i+1}^p, z_{i+1}^p) + \sum_{j=0}^{i} a_{j,i+1}F_3(t_j, x_j, y_j, z_j)\Big], \end{cases}$$

which is an iterative technique to solve this fractional problem.

4. **Robust control for variable order time fractional financial system**

A financial system can be defined from global, regional, or firm-specific points of view. It is defined as a set of procedures that are implemented in order to track financial activities. The financial system is, on a regional scale, the system enabling lenders and borrowers to exchange funds. Such a system covers financial transactions and money exchanged between investors, lenders, and borrowers. Financial systems are composed of intricate and complicated models depicting financial services, institutions, and markets that connect depositors with investors.

A firm's financial system is the set of procedures that are implemented for the purpose of monitor the companys financial activities. The global financial system is, in principle, a broader regional system involving all financial institutions, borrowers, and lenders on a global economic scope.

The financial system of different levels is composed of multiple components:

Within a firm, the financial system encompasses all aspects of finances. For example, it would include accounting measures, revenue, and expense schedules, wages, and balance sheet verification. Regional financial systems would include banks and other financial institutions, financial markets, financial services in a global view. Financial systems would include the international monetary fund, central banks, world bank and major banks that practice overseas lending.

Let us to consider the system [1, 7]

$$\begin{cases} ABC_0D_t^{q(t)}x(t) = z(t) + (y(t) - a)x(t), \\ ABC_0D_t^{q(t)}y(t) = 1 - by(t) - x(t)^2, \\ ABC_0D_t^{q(t)}z(t) = -x(t) - cz(t), \end{cases} \tag{4.1}$$

where the variable x denotes the interest rate, y is the investment demand, and z represents the price index. The parameters a, b, and c denotes the savings amount, cost per investment, and the elasticity of demand of commercial markets, respectively:

$$\begin{cases} ABC_0D_t^{q(t)}x(t) = z(t) + (y(t) - a)x(t) + u_x, \\ ABC_0D_t^{q(t)}y(t) = 1 - by(t) - x(t)^2 + u_y, \\ ABC_0D_t^{q(t)}z(t) = -x(t) - cz(t) + u_z. \end{cases} \tag{4.2}$$

The aim of robust control is to suppress the chaotic behavior in the systems. We define $\overline{x}$, $\overline{y}$, and $\overline{z}$ as equilibrium points of the auxiliary systems.

$$\begin{cases} ABC_0D_t^{q(t)}\overline{x}(t) = \overline{z}(t) + (\overline{y}(t) - a)\overline{x}(t), \\ ABC_0D_t^{q(t)}\overline{y}(t) = 1 - b\overline{y}(t) - \overline{x}(t)^2, \\ ABC_0D_t^{q(t)}\overline{z}(t) = -\overline{x}(t) - c\overline{z}(t). \end{cases} \tag{4.3}$$

Now, we define the control errors as

$$\begin{cases} e_x(t) = x(t) - \overline{x}(t), \\ e_y(t) = y(t) - \overline{y}(t), \\ e_z(t) = z(t) - \overline{z}(t). \end{cases} \tag{4.4}$$

Then

$$\begin{cases} ABC_0D_t^{q(t)}e_x(t) = e_z(t) + e_y(t)(e_x(t) + \overline{x}) - ae_x(t) + u_x, \\ ABC_0D_t^{q(t)}e_y(t) = 1 - be_y(t) + e_x(t)(e_x(t) + \overline{x}) + u_y, \\ ABC_0D_t^{q(t)}e_z(t) = -e_x(t) - ce_z(t) + u_z. \end{cases} \tag{4.5}$$

As was previously mentioned, the chaos behavior will be suppressed in the systems. Then equilibrium points are defined equal to zero i.e. $\overline{x} = \overline{y} = \overline{z} = 0$. Equation (4.5) simplifies to

$$\begin{cases} ABC_0D_t^{q(t)}e_x(t) = e_z(t) + e_x(t)e_y(t) - ae_x(t) + u_x, \\ ABC_0D_t^{q(t)}e_y(t) = 1 - be_y(t) + e_x(t)^2 + u_y, \\ ABC_0D_t^{q(t)}e_z(t) = -e_x(t) - ce_z(t) + u_z. \end{cases} \tag{4.6}$$

From (4.6), the control law is defined as follows:

$$\begin{cases} u_x = -e_z(t) - e_x(t)e_y(t) + ae_x(t) - k_xe_x(t), \\ u_y = -1 + be_y(t) - e_x(t)^2 - k_ye_y(t), \\ u_z = e_x(t) + ce_z(t) - k_ze_z(t), \end{cases} \tag{4.7}$$

where $k_x, k_y, k_z > 0$.

Theorem 4.1. *The system* (4.1) *is asymptotically stable if the control is defined as* (4.7).

Proof. The stability of the controller (4.7) can be proved using the following Lyapanov function

$$V(t) = \frac{1}{2}\Big(e_x^2(t) + e_y^2(t) + e_z^2(t)\Big).$$

Then, the derivative of the Lyapanov function is given by

$$\begin{aligned} ABC_0D_t^{q(t)}V(t) = e_x(t)ABC_0D_t^{q(t)}e_x(t) + e_y(t)ABC_0D_t^{q(t)}e_y(t) \\ + e_z(t)ABC_0D_t^{q(t)}e_z(t). \end{aligned} \tag{4.8}$$

From (4.6) and (4.7) the dynamic of each control error can be defined as

$$\begin{cases} ABC_0D_t^{q(t)}e_x(t) = e_z(t) + e_x(t)e_y(t) - ae_x(t) - e_z(t) - e_x(t)e_y(t) \\ \qquad\qquad +ae_x(t) - k_xe_x(t), \\ ABC_0D_t^{q(t)}e_y(t) = 1 - be_y(t) + e_x(t)^2 + -1 + be_y(t) - e_x(t)^2 - k_ye_y(t), \\ ABC_0D_t^{q(t)}e_z(t) = -e_x(t) - ce_z(t) + e_x(t) + ce_z(t) - k_ze_z(t). \end{cases} \tag{4.9}$$

Therefore,

$$\begin{cases} ABC_0D_t^{q(t)}e_x(t) = -k_xe_x(t), \\ ABC_0D_t^{q(t)}e_y(t) = -k_ye_y(t), \\ ABC_0D_t^{q(t)}e_z(t) = -k_ze_z(t). \end{cases} \tag{4.10}$$

Substituting (4.10) into (4.8), we get

$$ABC_0D_t^{q(t)}V(t) = -k_xe_x^2(t) - k_ye_y^2(t) - k_ze_z^2(t) \leq 0. \tag{4.11}$$

The fact that $k_x, k_y, k_z > 0$ assures that the derivative of the Lyapunov function will always be negative or equal to zero causing asymptomatic stability. □

5. **Simulation results**

In this section, we propose a generalization of the numerical scheme for the fractional financial chaotic system in Atangana-Baleanu-Caputo fractional derivatives with variable order $q(t)$. One of the initial motivations of our fractional control is to enhance the flexibility of performance adjustment.

The system (4.1), due to the implementation of the robust controller proposed with k_y, changes to

$$\begin{cases} ABC_0D_t^{q(t)}x(t) = z(t) + (y(t) - a)x(t), \\ ABC_0D_t^{q(t)}y(t) = 1 - by(t) - x(t)^2 - k_y y(t), \\ ABC_0D_t^{q(t)}z(t) = -x(t) - cz(t). \end{cases}$$

We obtain the numerical results with respect to $x_0 = 0.1$, $y_0 = 0.2$ and $z_0 = 0.3$. In Figure 1, the phase diagram is plotted for the fixed differential order $q(t) = 1$ and robust controller with the same derivative order is plotted in Figure 2 ($k_y = 80$). The system simulation was performed over 100 seconds.

The phase diagram and robust controller with $k_y = 45$ are plotted with respect to $q(t) = 0.99 - \dfrac{0.01}{100}t$ in Figures 3 and 4, respectively. From this Figure, it is clear to see that the Adams-Bashforth-Moulton method can solve variable-order fractional differential equation simply and effectively. Moreover, in comparison with Figure 1 and 2, the results of system can be found that the order of derivative effects strongly.

Finally, the phase diagram and robust controller with $k_y = 35$ are plotted with respect to $q(t) = 0.97 + 0.03\cos\left(\dfrac{t}{10}\right)$ in Figures 5 and 6, respectively. It is found that the variable-order fractional financial system can exhibit the stable equilibrium point, quasi-periodic trajectory, and chaotic motion with different order functions. The phase portrait and largest Lyapunov exponent are used to identify the dynamics of the financial system. The positive largest Lyapunov exponent implies that the financial system will generate chaotic motion. The dynamics of the financial system is somewhat different from classical financial system whose order functions remain one.

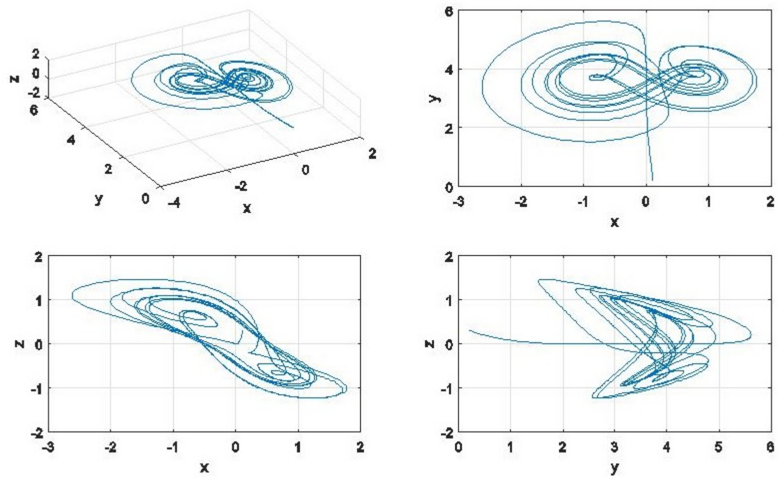

FIGURE 1. Phase diagram of the system for the order $q(t) = 1$

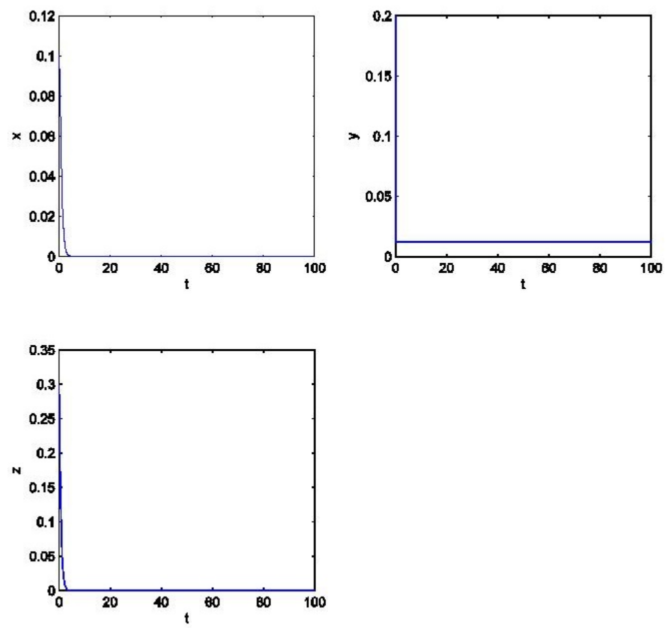

FIGURE 2. Proposed robust controller of the system for the order $q(t) = 1$

The above analysis implies that designers could obtain the satisfied turbofan dynamic behavior and economical cost by determining the appropriate fractional control.

6. **Conclusion**

A numerical scheme based on Adams method was proposed to get a numerical solution for variable-order fractional financial systems. The alternative numerical method was introduced to enhance the limitations of the variable-order Adams Bashforth method. The method is accurate, efficient, and direct. Furthermore, unlike

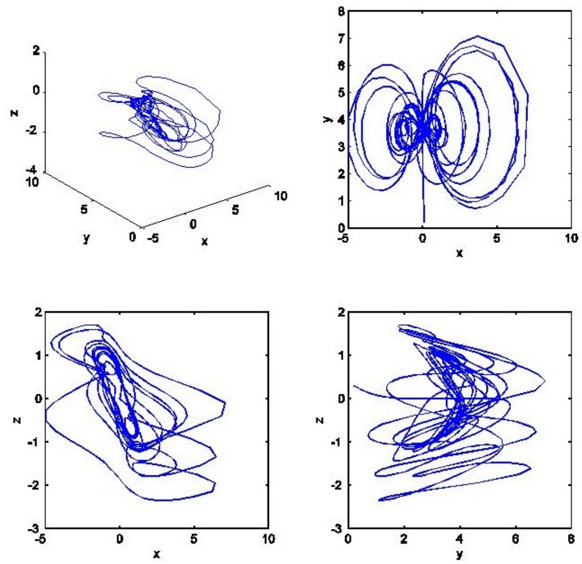

FIGURE 3. Phase diagram of the system for the order $q(t) = 0.99(0.01/100)t$

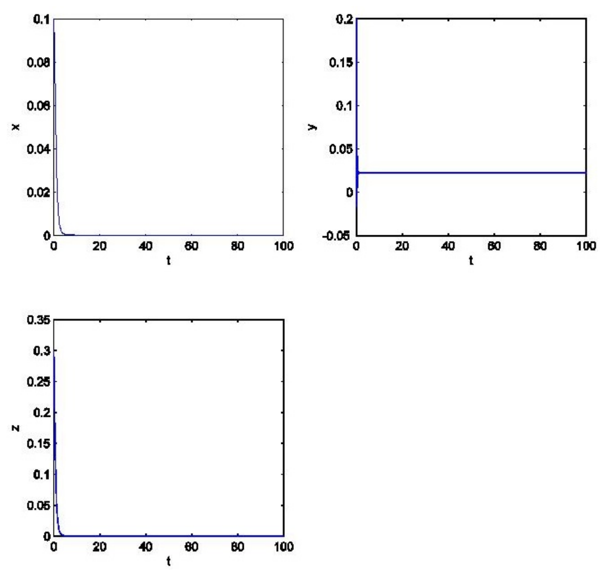

FIGURE 4. Proposed robust controller of the system for the order $q(t) = 0.99(0.01/100)t$

the Adams Bashforth method, a small step of discretization is not needed; thus, reducing the computational effort. Numerical examples with different variable-orders have been presented to demonstrate the effectiveness of the method. The application of the proposed algorithm to solve variable-order nonlinear fractional differential equations will be considered in future works. The main contribution of this control was: (A) the fractional decentralized robust control can guarantee the interconnection dynamic system uniform boundedness and ultimate uniform boundedness regardless

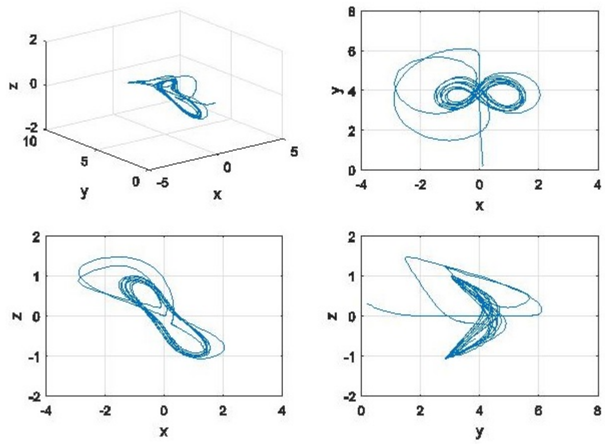

FIGURE 5. Phase diagram of the system for the order $q(t) = 0.97 + 0.03\cos(t/10)$

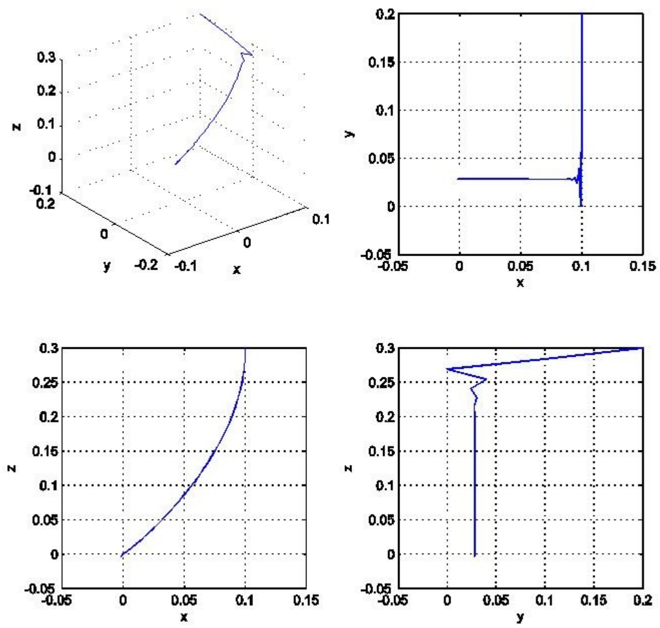

FIGURE 6. Proposed robust controller of the system for the order $q(t) = 0.97 + 0.03\cos(t/10)$

of the nonlinear uncertainty. The calculation of control inputs only needs the system states and avoids the requirement of nominal states; (B) the control has a more flexible and general structure which was one of the motivations behind of our work.

REFERENCES

[1] M. S. Abd-Elouahab, N. E. Hamri and J. Wang, Chaos control of a fractional-order financial system, *Mathematical Problems in Engineering*, 2010.

[2] H. Aminikhah, A. H. Refahi Sheikhani, T. Houlari and H. Rezazadeh, Numerical solution of the distributed-order fractional Bagley-Torvik equation, *IEEE/CAA Journal of Automatica Sinica*, 6(3):760–765, 2019.

[3] H. Aminikhah, A. H. Refahi Sheikhani and H. Rezazadeh, Stability analysis of linear distributed order system with multiple time delays, *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 77(2):207–218, 2015.

[4] H. Aminikhah, A. H. Refahi Sheikhani and H. Rezazadeh, Approximate analytical solutions of distributed order fractional Riccati differential equation, *Ain Shams Engineering Journal*, 9(4):581–588, 2018.

[5] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation, *Journal of Computational Physics*, 293:104–114, 2015.

[6] A. Atangana, Non validity of index law in fractional calculus: A fractional differential operator with markovian and non-markovian properties, *Physica A: Statistical Mechanics and its Applications*, 505:688–706, 2018.

[7] A. Atangana and D. Baleanu, New fractional derivatives with nonlocal and non-singular karnel: *Theory and application to heat transfer model, Thermal Science*, 20, 2016.

[8] A. Atangana and J.F. Botha, A generalized groundwater flow equation using the concept of variable-order derivative, *Boundary Value Problems*, 2013(1):1–11, 2013.

[9] A. Bhrawy and M. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation, *Nonlinear Dynamics*, 80(1-2):101–116, 2015.

[10] V. Daftardar-Gejji, Y. Sukale and S. Bhalekar, A new predictor-corrector method for fractional differential equations, *Applied Mathematics and Computation*, 244:158–182, 2014.

[11] K. Diethelm, N. J. Ford and A. D. Freed, Apredictor-corrector approach for the numerical solution of fractional differential equations, *Nonlinear Dynamics*, 29(1-4):3–22, 2002.

[12] K. Diethelm, N. J. Ford and A. D. Freed, Detailed error analysis for a fractional adams method, *Numerical algorithms*, 36(1):31–52, 2004.

[13] P. Gholamin and A. H. Refahi Sheikhani, A new three-dimensional chaotic systern: dynamical properties and simulation, *Chinese Journal of Physics*, 55(4):1300–1309, 2017.

[14] P. Gholamin and A. H. Refahi Sheikhani, Dynamical anlysis of a new three-dimensional fractional chaotic system, *Paramana Journal of Physics*, 92(6):1–14, 2019.

[15] M. J. Hai and C. Y. Shu, Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (i), *Applied Mathematics and Mechanics*, 22(11):1240–1251, 2001.

[16] M. J. Hai and C. Y. Shu, Study for the bifurcation topological structure and the global complicated character of a kind of nonlinear finance system (ii), *Applied Mathematics and Mechanics*, 22(12):1375–1382, 2001.

[17] M. S. Hashemi and A. Akgül, Solitary wave solutions of time-space nonlinear fractional schrödingers equation. Two analytical approaches, *Journal of Computational and Applied Mathematics*, 339:147–160, 2018.

[18] J. Hristov, The non-linear dodson diffusion equation: Approximate solutions and beyond with formalistic fractionalization, *Mathematics in Natural Science*, 1(1):1–17, 2017.

[19] A. Jiahui and C. Pengyu, Uniqueness of solutions to initial value problem of fractional differential equations of variable-order, *Dynamic Systems and Applications*, 28(3):607–623, 2019.

[20] Y.M. Guo, Y. Zhao, Y.M. Zhou, Z.B. Xiao and X.J. Yang, On the local fractional lwr model in fractal traffic flows in the entropy condition, *Mathematical Methods in the Applied Sciences*, 40(17):6127–6132, 2017.

[21] S. Ma, Y. Xu and W. Yue, Numerical solutions of a variable-order fractional financial system, *Journal of Applied Mathematics*, 2012.

[22] M. Mashoof, A. H. Refahi Sheikhani and H. Saberi Najafi, Stability Analysis of Distributed-Order Hilfer-Prabhakar Systems Based on Inertia Theory, *Mathematical Notes*, 104(1-2):74–85, 2018.

[23] M. Mashoof, A. H. Refahi Sheikhani and H. Saberi Najafi, Stability analysis of distributed order Hilfer-Prabhakar differential equations, *Hacettepe Journal of Mathematics and Statistics*, 47(2):299-315, 2018.

[24] K. M. Owolabi, Mathematical modelling and analysis of two-component system with caputo fractional derivative order, *Chaos, Solitons* & *Fractals*, 103:544-554, 2017.

[25] A. H. Refahi Sheikhani and M. Mashoof, Numerical solution of fractional differential equation by wavelets and hybrid functions, *Boletim da Sociedade Paranaense de Matematica*, 36(2):231–244, 2018.

[26] A. U. Rehman, O. Khan, N. Ali and M. Pervaiz, Nonlinear robust control of a variable speed-wind turbine. In: Engineering and Emerging Technologies (ICEET), *International Conference IEEE*, 1–6, 2018.

[27] S. Samko, Fractional integration and differentiation of variable order: an overview, *Nonlinear Dynamics*, 71(4):653-662, 2013.

[28] J. Singh, D. Kumar, Z. Hammouch and A. Atangana, A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, *Applied Mathematics and Computation*, 316:504–515, 2018.

[29] J. Solís Pérez, J. Gómez-Aguilar and A. Atangana, Novel numerical method for solving variable-order fractional differential equations with power, exponential and mittag-leffler laws, *Chaos, Solitons* & *Fractals*, 114:175–185, 2018.

[30] M. N. Soorki and M. S. Tavazoei, Adaptive robust control of fractional-order swarm systems in the presence of model uncertainties and external disturbances, *IET Control Theory* & *Applications*, 12(7):961–969, 2018.

[31] H. Sun, W. Chen, C. Li and Y. Chen, Fractional differential models for anomalous diffusion, Physica A: *Statistical Mechanics and its Applications*, 389(14):2719–2724, 2010.

[32] H. Sun, H. Zhao, K. Huang, M. Qiu, S. Zhen and Y. H. Chen, A fuzzy approach for optimal robust control design of an automotive electronic throttle system, *IEEE Transactions on Fuzzy Systems*, 26(2):694–704, 2018.

[33] M. Toufik and A. Atangana, New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models, *The European Physical Journal Plus*, 132(10):1–16, 2017.

[34] B. Wang, J. Ding, F. Wu and D. Zhu, Robust finite-time control of fractional-order nonlin-earsystems via frequency distributed model, *Nonlinear Dynamics*, 85(4):2133–2142, 2016.

[35] J. Wang, Z. Duan, G. Wen and G. Chen, Distributed robust control of uncertain linearmulti-agent systems, *International Journal of Robust and Nonlinear Control*, 25(13):2162–2179, 2015.

[36] G. W. Wang and M. S. Hashemi, Lie symmetry analysis and soliton solutions of time-fractional $k(m, n)$ equation, *Pramana*, 88(1):1–6, 2017.

[37] C. Zuriga-Aguilar, J. Gómez-Aguilar, R. Escobar-Jiménez and H. Romero-Ugalde, Robust control for fractional variable-order chaotic systems with non-singular kernel, *The European Physical Journal Plus*, 133(1), 2018.