Neural, Parallel, and Scientific Computations 20 (2012) 415-436

A NEW HIGH ACCURACY VARIABLE MESH DISCRETIZATION FOR THE SOLUTION OF THE SYSTEM OF 2D NON-LINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

NIKITA SETIA AND R.K. MOHANTY[1]

Department of Mathematics
Faculty of Mathematical Sciences
University of Delhi, Delhi-110007, India
rmohanty@maths.du.ac.in

Abstract: In this paper, we develop an $O(k^2 + k^2 h_l + h_l^3)$ nine-point compact off-step finite difference discretization for the solution of the system of two-dimensional non-linear elliptic equations subject to Dirichlet boundary conditions, by using variable mesh lengths h_l in x-direction and a constant mesh length k in y-direction. We use only three function evaluations. Further we discuss the conditions for the convergence of the iterative methods applied to the system of difference equations so framed for the steady state 2D convection-diffusion equation. Numerical illustrations of some benchmark problems including 2D non-linear convection equation and 2D steady-state Navier-stokes equations of motion are provided to depict the efficiency of the method.

Keywords: Variable mesh; Convection-diffusion equation; Off-step discretization; Non linear convection equation; Navier-stokes equations of motion.

1. INTRODUCTION

We consider the following system of two dimensional non-linear elliptic boundary value problems

$$u_{xx}^{(i)} + u_{yy}^{(i)} = f^{(i)}(x, y, u^{(1)}, u^{(2)}, ..., u^{(n)}, u_x^{(1)}, u_x^{(2)}, ..., u_x^{(n)}, u_y^{(1)}, u_y^{(2)}, ..., u_y^{(n)}), \quad (1)$$

defined in a bounded region $\Omega = \{(x, y) \mid 0 < x, y < 1\}$ with boundary $\partial\Omega$, such that

$$u^{(i)}(x, y) = u_0^{(i)}(x, y)\,; \quad (x, y) \in \partial\Omega, \quad (2)$$

where throughout this paper, i varies from 1, 2,…, n, where $n > 0$ is a positive integer.

We assume that for $(x, y) \in \Omega$ and j =1,2,…,n,

Received May 2, 2011

a) each $f^{(i)}(x, y, u^{(1)}, u^{(2)}, ..., u^{(n)}, u_x^{(1)}, u_x^{(2)}, ..., u_x^{(n)}, u_y^{(1)}, u_y^{(2)}, ..., u_y^{(n)})$ is continuous,

b) $\frac{\partial f^{(i)}}{\partial u^{(j)}}, \frac{\partial f^{(i)}}{\partial u_x^{(j)}}$ and $\frac{\partial f^{(i)}}{\partial u_y^{(j)}}$ exist and are continuous,

c) $\frac{\partial f^{(i)}}{\partial u^{(j)}} > 0$, $\left|\frac{\partial f^{(i)}}{\partial u_x^{(j)}}\right| \le G_{(j)}^{(i)}$ and $\left|\frac{\partial f^{(i)}}{\partial u_y^{(j)}}\right| \le H_{(j)}^{(i)}$.

where $G_{(j)}^{(i)}$ and $H_{(j)}^{(i)}$ are positive constants. These conditions guarantee the existence and uniqueness of the solution of the above system of equations (Jain *et al*, 1991). Further we assume that each $u^{(i)} \in C^6(\Omega)$, where $C^m(\Omega)$ denotes the class of functions of x and y whose partial derivatives upto order m are continuous in Ω.

The second order non-linear elliptic partial differential equations (PDEs) occur in the formulation of many applied problems in physics and engineering. There has been a considerable interest by many authors in the development of compact finite difference schemes for the solution of the linear as well as the non-linear elliptic boundary value problems (Yavneh, 1997), (Zhang, 1997, 1998), (Spotz and Carey, 1995), (Sakurai *et al*, 2002), (Jain *et al*, 1994), (Saldanha, 2001), (Ananthakrishnaiah and Saldanha, 1995). The standard central difference schemes though are simple to apply and yield second order accuracy, they usually fail when applied to singular perturbation problems, specially when the perturbation parameter ε (say) is small. (Jain *et al*, 1989) developed a compact fourth order discretization for elliptic equations with non-linear first derivative terms and constant coefficients using only 9 grid points, which was further extended to the system of elliptic PDEs with variable coefficients by (Jain *et al*, 1991) and (Mohanty, 1997). These schemes used equal mesh sizes in both the coordinate directions. (Mohanty *et al*, 2006) proposed an unequal mesh 9-point fourth order scheme for the solution of non-linear elliptic PDEs with variable coefficients. However, all these schemes required modification at the points of singularity. In this regard, (Mohanty and Singh, 2006) developed a high order arithmetic average discretization for the singularly perturbed 2D nonlinear problems. All the above schemes (Mohanty *et al*, 2006), (Jain *et al*, 1989), (Jain *et al*, 1991), (Mohanty, 1997) and (Mohanty and Singh, 2006) were uniform mesh schemes and required five function evaluations. Even the high order schemes would fail to give accurate results when the perturbation factor ε is small. This is because using a constant mesh length, attaining convergence at all mesh points uniformly in ε becomes difficult. For instance, we consider the one dimensional steady state convection diffusion equation

$$\varepsilon u_{xx} = u_x + f(x) \tag{3}$$

subject to the boundary conditions $u(0) = \alpha$ and $u(1) = \beta$. This equation models the temperature $u(x)$ of a fluid flowing through a pipe with a constant velocity, say a (called the convective velocity) where fluid has constant heat diffusion coefficient, say κ. Then the perturbation parameter ε is given by the equation $\varepsilon = \kappa / a$. Physically, we should expect difficulties in case where the convective velocity *a* overwhelms the diffusivity factor κ, i.e. when $\varepsilon << 1$, since in this case it would be very difficult to maintain a fixed temperature (β here) at the outflow of the tube. Mathematically, we expect trouble as $\varepsilon \to 0$ because in the limit $\varepsilon = 0$, the above equation (3) reduces to a first order equation $u_x + f(x) = 0$ which allows only one boundary condition, rather than two. However, for $\varepsilon > 0$, no matter how small ε is, we have a second order equation that needs two conditions. Thus as $\varepsilon \to 0$, the solution tends towards a discontinuous function that jumps to the value β at the last possible moment. This region of rapid transition is called the *boundary layer*. It is from here that the need of choosing a variable mesh arises. In the past, some variable mesh methods have been developed for the solution of singularly perturbed two point boundary value problems by (Jain *et al*, 1983), (Mohanty, 2005) and (Kadalbajoo and Kumar, 2010). Recently, (Mohanty and Setia, 2012) have proposed a new nine point fourth order accurate numerical method based on off-step discretization on a constant mesh for the solution of the system of two dimensional nonlinear elliptic partial differential equations. In this paper, we design a high order variable mesh off-step discretization for the solution of the system of two-dimensional non-linear elliptic PDEs (1), using a constant mesh length *k* in *y*-direction and variable mesh lengths h_l in *x*-direction and 9 grid points of a single computational cell (see Fig.1). This method not only gives accurate results for small values of perturbation parameter, but is also relatively simple to apply as it requires only three function evaluations and can be directly applied to singular problems as well, without any modification.

This paper is organized as follows: In Section 2, the $O(k^2 + k^2 h_l + h_l^3)$ compact off-step discretization is described for the corresponding scalar elliptic boundary value problem. In Section 3, this discretization is derived and extended to the system of equations (1). In Section 4, we discuss the conditions for the convergence of the iterative methods to be applied to solve the tri-block-diagonal system of difference equations so obtained. In Section 5, we give numerical examples to illustrate our method. Section 6 contains some concluding remarks on this paper.

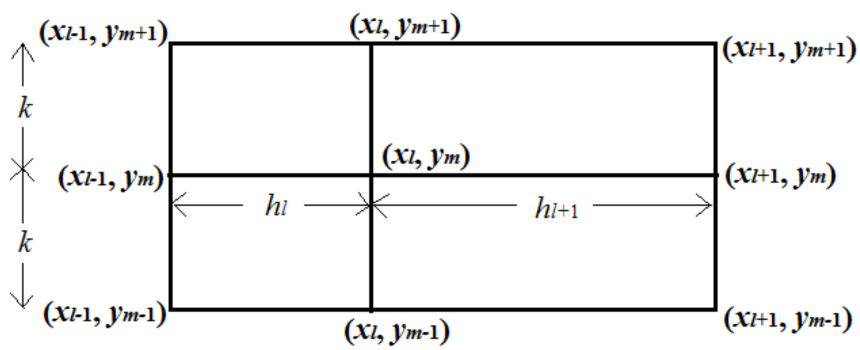


Figure 1: Single Computational Cell

2. DESCRIPTION OF THE METHOD

For simplicity, we first consider the following two dimensional non-linear elliptic PDE

$$u_{xx} + u_{yy} = f(x, y, u, u_x, u_y) \tag{4}$$

defined in Ω subject to

$$u(x, y) = u_0(x, y)\,; \qquad (x, y) \in \partial\Omega \tag{5}$$

We discretize the region Ω with a rectangular mesh by taking a constant mesh length $k > 0$ in y - direction and variable mesh lengths h_l in x – direction so that each grid point is given by (x_l, y_m) where $0 = x_0 < x_1 < x_2 < \ldots < x_{N+1} = 1$, $h_{l+1} = x_{l+1} - x_l$, for l = 0(1)N with the mesh ratio $\sigma_l = (h_{l+1}/h_l)$ > 0, for l = 1(1)N, and $y_m = mk$, for m = 0(1)M+1, where N and M are positive integers such that $(M+1)k = 1$.

Further, let $U_{l,m}$ and $u_{l,m}$ be the exact and approximate solution values of $u(x, y)$, respectively, at the grid point (x_l, y_m).

At each grid point (x_l, y_m), equation (4) may be written as

$$U_{xxl,m} + U_{yyl,m} = f(x_l, y_m, U_{l,m}, U_{xl,m}, U_{yl,m}) \equiv f_{l,m} \tag{6}$$

We set the following approximations based on the approach of (Chawla and Shivakumar, 1996).

$$\overline{U}_{l\pm\frac{1}{2},m} = \frac{1}{2}\left(U_{l\pm1,m} + U_{l,m}\right) \tag{7.1}$$

$$\overline{U}_{xl,m} = \frac{U_{l+1,m} + (\sigma_l^2 - 1)U_{l,m} - \sigma_l^2 U_{l-1,m}}{h_l \sigma_l (1+\sigma_l)} \tag{7.2}$$

$$\overline{U}_{xl+\frac{1}{2},m} = \frac{1}{\sigma_l h_l}\left(U_{l+1,m} - U_{l,m}\right) \tag{7.3}$$

$$\overline{U}_{xl-\frac{1}{2},m} = \frac{1}{h_l}\left(U_{l,m} - U_{l-1,m}\right) \tag{7.4}$$

$$\overline{U}_{yl,m} = \frac{1}{2k}\left(U_{l,m+1} - U_{l,m-1}\right) \tag{7.5}$$

$$\overline{U}_{yl\pm\frac{1}{2},m} = \frac{1}{4k}\left(U_{l\pm1,m+1} - U_{l\pm1,m-1} + U_{l,m+1} - U_{l,m-1}\right) \tag{7.6}$$

$$\overline{U}_{yl\pm1,m} = \frac{1}{2k}\left(U_{l\pm1,m+1} - U_{l\pm1,m-1}\right) \tag{7.7}$$

$$\overline{U}_{xxl,m} = \frac{2\left[U_{l+1,m} - (1+\sigma_l)U_{l,m} + \sigma_l U_{l-1,m}\right]}{h_l^2 \sigma_l (1+\sigma_l)} \tag{7.8}$$

$$\overline{U}_{yyl,m} = \frac{1}{k^2}\left(U_{l,m+1} - 2U_{l,m} + U_{l,m-1}\right) \tag{7.9}$$

$$\overline{U}_{yyl\pm1,m} = \frac{1}{k^2}\left(U_{l\pm1,m+1} - 2U_{l\pm1,m} + U_{l\pm1,m-1}\right) \tag{7.10}$$

Further, we define

$$\overline{f}_{l\pm\frac{1}{2},m} = f(x_{l\pm\frac{1}{2}}, y_m, \overline{U}_{l\pm\frac{1}{2},m}, \overline{U}_{xl\pm\frac{1}{2},m}, \overline{U}_{yl\pm\frac{1}{2},m}) \tag{8}$$

Let

$$\overline{\overline{U}}_{l,m} = U_{l,m} - \frac{5h_l^2}{8}\left(\frac{1+\sigma_l^3}{1+\sigma_l}\right)\left(\overline{f}_{l+\frac{1}{2},m} + \overline{f}_{l-\frac{1}{2},m}\right) + h_l^2\left(\frac{1+\sigma_l^3}{1+\sigma_l}\right)\overline{U}_{xxl,m} + \frac{5h_l^2}{4}\left(\frac{1+\sigma_l^3}{1+\sigma_l}\right)\overline{U}_{yyl,m} \tag{9.1}$$

$$\overline{\overline{U}}_{xl,m} = \overline{U}_{xl,m} + h_l\left[\frac{(1+\sigma_l^3)}{12(1+\sigma_l)^2} + \frac{\sigma_l}{6(1+\sigma_l)}\right]\left[\left(\overline{U}_{yyl+1,m} - \overline{U}_{yyl-1,m}\right) - 2\left(\overline{f}_{l+\frac{1}{2},m} - \overline{f}_{l-\frac{1}{2},m}\right)\right] \tag{9.2}$$

$$\overline{\overline{U}}_{yl,m} = \overline{U}_{yl,m} - \frac{(1+\sigma_l^3)}{2\sigma_l(1+\sigma_l)^2}\left[\overline{U}_{yl+1,m} - (1+\sigma_l)\overline{U}_{yl,m} + \sigma_l \overline{U}_{yl-1,m}\right] \tag{9.3}$$

Finally, we define

$$\overline{\overline{f}}_{l,m} = f(x_l, y_m, \overline{\overline{U}}_{l,m}, \overline{\overline{U}}_{xl,m}, \overline{\overline{U}}_{yl,m}) \tag{10}$$

Then, at each internal grid point (x_l, y_m), the partial differential equation (4) is discretized by the following finite difference scheme :

$$I_1\left(U_{l+1,m+1} + U_{l+1,m-1}\right) + I_2\left(U_{l-1,m+1} + U_{l-1,m-1}\right) + I_3\left(U_{l,m+1} + U_{l,m-1}\right) + I_4 U_{l+1,m} + I_5 U_{l-1,m} + I_6 U_{l,m}$$

$$= \frac{\sigma_l h_l^2}{3}\left[\sigma_l \overline{f}_{l+\frac{1}{2},m} + \left(\frac{1+\sigma_l}{2}\right)\overline{\overline{f}}_{l,m} + \overline{f}_{l-\frac{1}{2},m}\right] + \overline{T}_{l,m}, \quad [l = 1(1)N,\ m = 1(1)M] \tag{11}$$

where

$\overline{T}_{l,m} = O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)$, and

$$I_1 = \frac{\sigma_l^2 h_l^2}{6k^2} - \frac{(1+\sigma_l^3)h_l^2}{12k^2(1+\sigma_l)}, \qquad I_2 = \frac{\sigma_l h_l^2}{6k^2} - \frac{\sigma_l(1+\sigma_l^3)h_l^2}{12k^2(1+\sigma_l)},$$

$$I_3 = \frac{\sigma_l(1+\sigma_l)h_l^2}{3k^2} + \frac{(1+\sigma_l^3)h_l^2}{12k^2}, \qquad I_4 = 1 - \frac{\sigma_l^2 h_l^2}{3k^2} + \frac{(1+\sigma_l^3)h_l^2}{6k^2(1+\sigma_l)},$$

$$I_5 = \sigma_l - \frac{\sigma_l h_l^2}{3k^2} + \frac{\sigma_l(1+\sigma_l^3)h_l^2}{6k^2(1+\sigma_l)}, \qquad I_6 = -\left[(1+\sigma_l) + \frac{2\sigma_l(1+\sigma_l)h_l^2}{3k^2} + \frac{(1+\sigma_l^3)h_l^2}{6k^2}\right].$$

We note that the difference method (11) is a nine-point formulae which can be conveniently expressed in the matrix form **Au = B**, where the coefficient matrix **A** is tri-block-diagonal. This system of difference equations so obtained can be solved by the Newton-Raphson method for the non-linear case and by Gauss-Siedal or Jacobi iteration method for the linear case [see (Kelly, 1995), (Varga, 2000), (Saad, 2003), (Hageman & Young 2004)].

3. DERIVATION PROCEDURE

At each grid point (x_l, y_m), let us denote

$$\alpha_{l,m} = \left(\frac{\partial f}{\partial U}\right)_{l,m}, \qquad \beta_{l,m} = \left(\frac{\partial f}{\partial U_x}\right)_{l,m}, \qquad \gamma_{l,m} = \left(\frac{\partial f}{\partial U_y}\right)_{l,m}$$

Simplifying the approximations (7.1) - (7.10), we obtain

$$\overline{U}_{l+\frac{1}{2},m} = U_{l+\frac{1}{2},m} + \frac{\sigma_l^2 h_l^2}{8} U_{xxl,m} + O(h_l^3) \tag{12.1}$$

$$\overline{U}_{l-\frac{1}{2},m} = U_{l-\frac{1}{2},m} + \frac{h_l^2}{8} U_{xxl,m} + O(h_l^3) \tag{12.2}$$

$$\overline{U}_{xl,m} = U_{xl,m} + \frac{\sigma_l h_l^2}{6} U_{xxxl,m} + O(h_l^3) \tag{12.3}$$

$$\overline{U}_{xl+\frac{1}{2},m} = U_{xl+\frac{1}{2},m} + \frac{\sigma_l^2 h_l^2}{24} U_{xxxl,m} + O(h_l^3) \tag{12.4}$$

$$\overline{U}_{xl-\frac{1}{2},m} = U_{xl-\frac{1}{2},m} + \frac{h_l^2}{24} U_{xxxl,m} + O(h_l^3) \tag{12.5}$$

$$\overline{U}_{yl,m} = U_{yl,m} + O(k^2) \tag{12.6}$$

$$\overline{U}_{yl+\frac{1}{2},m} = U_{yl+\frac{1}{2},m} + \frac{\sigma_l^2 h_l^2}{8} U_{xxyl,m} + O(k^2 + k^2 h_l + h_l^3) \tag{12.7}$$

$$\overline{U}_{yl-\frac{1}{2},m} = U_{yl-\frac{1}{2},m} + \frac{h_l^2}{8} U_{xxyl,m} + O(k^2 + k^2 h_l + h_l^3) \tag{12.8}$$

$$\overline{U}_{yl+1,m} = U_{yl+1,m} + O(k^2 + k^2 h_l) \tag{12.9}$$

$$\overline{U}_{yl-1,m} = U_{yl-1,m} + O(k^2 + k^2 h_l) \tag{12.10}$$

$$\overline{U}_{xxl,m} = U_{xxl,m} + O(h_l) \tag{12.11}$$

$$\overline{U}_{yyl,m} = U_{yyl,m} + O(k^2) \tag{12.12}$$

$$\overline{U}_{yyl+1,m} = U_{yyl+1,m} + O(k^2 + k^2 h_l) \tag{12.13}$$

$$\overline{U}_{yyl-1,m} = U_{yyl-1,m} + O(k^2 + k^2 h_l) \tag{12.14}$$

Further by Taylor series expansion, we first obtain

$$I_1\left(U_{l+1,m+1} + U_{l+1,m-1}\right) + I_2\left(U_{l-1,m+1} + U_{l-1,m-1}\right) + I_3\left(U_{l,m+1} + U_{l,m-1}\right) + I_4 U_{l+1,m} + I_5 U_{l-1,m} + I_6 U_{l,m}$$

$$= \frac{\sigma_l h_l^2}{3}\left[\sigma_l f_{l+\frac{1}{2},m} + \left(\frac{1+\sigma_l}{2}\right) f_{l,m} + f_{l-\frac{1}{2},m}\right] + O(k^2 h_l^2 + h_l^5), \quad [l = 1(1)N,\ m = 1(1)M]. \tag{13}$$

With the help of approximations (12.1) – (12.8), we obtain

$$\overline{f}_{l+\frac{1}{2},m} = f_{l+\frac{1}{2},m} + \frac{\sigma_l^2 h_l^2}{24} T_1 + O(k^2 + k^2 h_l + h_l^3 + k^2 h_l^2 + h_l^4) \tag{14.1}$$

$$\overline{f}_{l-\frac{1}{2},m} = f_{l-\frac{1}{2},m} + \frac{h_l^2}{24} T_1 + O(k^2 + k^2 h_l + h_l^3 + k^2 h_l^2 + h_l^4) \tag{14.2}$$

where

$$T_1 = 3U_{xxl,m}\alpha_{l,m} + U_{xxxl,m}\beta_{l,m} + 3U_{xxyl,m}\gamma_{l,m}.$$

Now, let

$$\overline{\overline{U}}_{l,m} = U_{l,m} + a_1 h_l^2 \overline{f}_{l+\frac{1}{2},m} + a_2 h_l^2 \overline{f}_{l-\frac{1}{2},m} + a_3 h_l^2 \overline{U}_{xxl,m} + a_4 h_l^2 \overline{U}_{yyl,m} \tag{15.1}$$

$$\overline{\overline{U}}_{xl,m} = \overline{U}_{xl,m} + b_1 h_l (\overline{f}_{l+\frac{1}{2},m} - \overline{f}_{l-\frac{1}{2},m}) + b_2 h_l (\overline{U}_{yyl+1,m} - \overline{U}_{yyl-1,m}) + b_3 h_l^2 \overline{U}_{xxl,m} + b_4 h_l^2 \overline{U}_{yyl,m} \tag{15.2}$$

$$\overline{\overline{U}}_{yl,m} = \overline{U}_{yl,m} + c\left[\overline{U}_{yl+1,m} - (1+\sigma_l)\overline{U}_{yl,m} + \sigma_l \overline{U}_{yl-1,m}\right] \tag{15.3}$$

where $a_q s$, $b_q s$ $(q = 1(1)4)$ and c are parameters to be suitably determined.

Now, with the help of (12.9)-(12.14), (14.1), (14.2), from (15.1)-(15.3), we obtain

$$\overline{\overline{U}}_{l,m} = U_{l,m} + \frac{h_l^2}{6} T_2 + O(k^2 h_l^2 + h_l^3) \tag{16.1}$$

$$\overline{\overline{U}}_{xl,m} = U_{xl,m} + \frac{h_l^2}{6} T_3 + O(k^2 h_l^2 + h_l^3) \tag{16.2}$$

$$\overline{\overline{U}}_{yl,m} = U_{yl,m} + \frac{h_l^2}{6} T_4 + O(k^2 + k^2 h_l^2 + h_l^3) \tag{16.3}$$

where

$$T_2 = 6(a_1 + a_2 + a_3)U_{xxl,m} + 6(a_1 + a_2 + a_4)U_{yyl,m},$$

$$T_3 = \left[\sigma_l + 3b_1(1+\sigma_l)\right]U_{xxxl,m} + 3(1+\sigma_l)(b_1 + 2b_2)U_{xyyl,m} + 6b_3 U_{xxl,m} + 6b_4 U_{yyl,m},$$

$$T_4 = 3c\sigma_l(1+\sigma_l)U_{xxyl,m}.$$

Now, by the help of the approximations (16.1)-(16.3), from (10), we obtain

$$\bar{\bar{f}}_{l,m} = f_{l,m} + \frac{h_l^2}{6} T_5 + O(k^2 + k^2 h_l^2 + h_l^3) \tag{17}$$

where

$$T_5 = T_2 \alpha_{l,m} + T_3 \beta_{l,m} + T_4 \gamma_{l,m} .$$

Using (13), (14.1), (14.2) and (17) in (11), we obtain

$$\bar{T}_{l,m} = -\left[\frac{\sigma_l (1+\sigma_l^3)}{72} T_1 + \frac{\sigma_l (1+\sigma_l)}{36} T_5 \right] h_l^4 + O(k^2 h_l^2 + k^2 h_l^3 + h_l^5) \tag{18}$$

Thus, for the proposed difference method (11) to be of $O(k^2 + k^2 h_l + h_l^3)$, we must have

$$\frac{(1+\sigma_l^3)}{72} T_1 + \frac{(1+\sigma_l)}{36} T_5 = 0 \tag{19}$$

Equating to zero the coefficients of $\alpha_{l,m}, \beta_{l,m}$ and $\gamma_{l,m}$ in equation (19), we obtain

$$a_1 = a_2 = -\frac{5}{8}\left(\frac{1+\sigma_l^3}{1+\sigma_l} \right), \qquad a_3 = \frac{1+\sigma_l^3}{1+\sigma_l}, \qquad a_4 = \frac{5}{4}\left(\frac{1+\sigma_l^3}{1+\sigma_l} \right), \qquad c = -\frac{(1+\sigma_l^3)}{2\sigma_l (1+\sigma_l)^2},$$

$$b_1 = -\left[\frac{(1+\sigma_l^3)}{6(1+\sigma_l)^2} + \frac{\sigma_l}{3(1+\sigma_l)} \right], \qquad b_2 = \frac{(1+\sigma_l^3)}{12(1+\sigma_l)^2} + \frac{\sigma_l}{6(1+\sigma_l)}, \qquad b_3 = b_4 = 0 .$$

The above values of $a_q s$, $b_q s$ $(q = 1(1)4)$ and c reduce $\bar{T}_{l,m} = O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)$ $[l = 1(1)N, m = 1(1)M]$ and thus we obtain the required difference scheme of $O(k^2 + k^2 h_l + h_l^3)$.

Now, we generalize our method as follows: For the system of differential equations (1) subject to the Dirichlet boundary conditions (2), we set the following approximations:

$$\bar{U}^{(i)}_{l\pm\frac{1}{2},m} = \frac{1}{2}\left(U^{(i)}_{l\pm1,m} + U^{(i)}_{l,m} \right) \tag{20.1}$$

$$\bar{U}^{(i)}_{xl,m} = \frac{U^{(i)}_{l+1,m} + (\sigma_l^2 - 1) U^{(i)}_{l,m} - \sigma_l^2 U^{(i)}_{l-1,m}}{h_l \sigma_l (1+\sigma_l)} \tag{20.2}$$

$$\bar{U}^{(i)}_{xl+\frac{1}{2},m} = \frac{1}{\sigma_l h_l}\left(U^{(i)}_{l+1,m} - U^{(i)}_{l,m} \right) \tag{20.3}$$

$$\bar{U}^{(i)}_{xl-\frac{1}{2},m} = \frac{1}{h_l}\left(U^{(i)}_{l,m} - U^{(i)}_{l-1,m} \right) \tag{20.4}$$

$$\overline{U}^{(i)}_{yl,m} = \frac{1}{2k}\left(U^{(i)}_{l,m+1} - U^{(i)}_{l,m-1}\right) \tag{20.5}$$

$$\overline{U}^{(i)}_{yl\pm\frac{1}{2},m} = \frac{1}{4k}\left(U^{(i)}_{l\pm1,m+1} - U^{(i)}_{l\pm1,m-1} + U^{(i)}_{l,m+1} - U^{(i)}_{l,m-1}\right) \tag{20.6}$$

$$\overline{U}^{(i)}_{yl\pm1,m} = \frac{1}{2k}\left(U^{(i)}_{l\pm1,m+1} - U^{(i)}_{l\pm1,m-1}\right) \tag{20.7}$$

$$\overline{U}^{(i)}_{xxl,m} = \frac{2\left[U^{(i)}_{l+1,m} - (1+\sigma_l)U^{(i)}_{l,m} + \sigma_l U^{(i)}_{l-1,m}\right]}{h_l^2 \sigma_l (1+\sigma_l)} \tag{20.8}$$

$$\overline{U}^{(i)}_{yyl,m} = \frac{1}{k^2}\left(U^{(i)}_{l,m+1} - 2U^{(i)}_{l,m} + U^{(i)}_{l,m-1}\right) \tag{20.9}$$

$$\overline{U}^{(i)}_{yyl\pm1,m} = \frac{1}{k^2}\left(U^{(i)}_{l\pm1,m+1} - 2U^{(i)}_{l\pm1,m} + U^{(i)}_{l\pm1,m-1}\right) \tag{20.10}$$

$$\overline{f}^{(i)}_{l\pm\frac{1}{2},m} = f^{(i)}(x_{l\pm\frac{1}{2}}, y_m, \overline{U}^{(1)}_{l\pm\frac{1}{2},m}, \overline{U}^{(2)}_{l\pm\frac{1}{2},m}, \ldots, \overline{U}^{(n)}_{l\pm\frac{1}{2},m}, \overline{U}^{(1)}_{xl\pm\frac{1}{2},m}, \overline{U}^{(2)}_{xl\pm\frac{1}{2},m}, \ldots, \overline{U}^{(n)}_{xl\pm\frac{1}{2},m}, \overline{U}^{(1)}_{yl\pm\frac{1}{2},m}, \overline{U}^{(2)}_{yl\pm\frac{1}{2},m}, \ldots, \overline{U}^{(n)}_{yl\pm\frac{1}{2},m}) \tag{20.11}$$

$$\overline{\overline{U}}^{(i)}_{l,m} = U^{(i)}_{l,m} - \frac{5h_l^2}{8}\left(\frac{1+\sigma_l^3}{1+\sigma_l}\right)\left(\overline{f}^{(i)}_{l+\frac{1}{2},m} + \overline{f}^{(i)}_{l-\frac{1}{2},m}\right) + h_l^2\left(\frac{1+\sigma_l^3}{1+\sigma_l}\right)\overline{U}^{(i)}_{xxl,m} + \frac{5h_l^2}{4}\left(\frac{1+\sigma_l^3}{1+\sigma_l}\right)\overline{U}^{(i)}_{yyl,m} \tag{20.12}$$

$$\overline{\overline{U}}^{(i)}_{xl,m} = \overline{U}^{(i)}_{xl,m} + h_l\left[\frac{(1+\sigma_l^3)}{12(1+\sigma_l)^2} + \frac{\sigma_l}{6(1+\sigma_l)}\right]\left[\left(\overline{U}^{(i)}_{yyl+1,m} - \overline{U}^{(i)}_{yyl-1,m}\right) - 2\left(\overline{f}^{(i)}_{l+\frac{1}{2},m} - \overline{f}^{(i)}_{l-\frac{1}{2},m}\right)\right] \tag{20.13}$$

$$\overline{\overline{U}}^{(i)}_{yl,m} = \overline{U}^{(i)}_{yl,m} - \frac{(1+\sigma_l^3)}{2\sigma_l(1+\sigma_l)^2}\left[\overline{U}^{(i)}_{yl+1,m} - (1+\sigma_l)\overline{U}^{(i)}_{yl,m} + \sigma_l \overline{U}^{(i)}_{yl-1,m}\right] \tag{20.14}$$

$$\overline{\overline{f}}^{(i)}_{l,m} = f^{(i)}(x_l, y_m, \overline{\overline{U}}^{(1)}_{l,m}, \overline{\overline{U}}^{(2)}_{l,m}, \ldots, \overline{\overline{U}}^{(n)}_{l,m}, \overline{\overline{U}}^{(1)}_{xl,m}, \overline{\overline{U}}^{(2)}_{xl,m}, \ldots, \overline{\overline{U}}^{(n)}_{xl,m}, \overline{\overline{U}}^{(1)}_{yl,m}, \overline{\overline{U}}^{(2)}_{yl,m}, \ldots, \overline{\overline{U}}^{(n)}_{yl,m}) \tag{20.15}$$

Then, it can be easily verified that at each grid point (x_l, y_m), $[l = 1(1)N, m = 1(1)M]$, the given system of nonlinear elliptic PDEs (1) is discretized by

$$I_1\left(U^{(i)}_{l+1,m+1} + U^{(i)}_{l+1,m-1}\right) + I_2\left(U^{(i)}_{l-1,m+1} + U^{(i)}_{l-1,m-1}\right) + I_3\left(U^{(i)}_{l,m+1} + U^{(i)}_{l,m-1}\right) + I_4 U^{(i)}_{l+1,m} + I_5 U^{(i)}_{l-1,m} + I_6 U^{(i)}_{l,m}$$
$$= \frac{\sigma_l h_l^2}{3}\left[\sigma_l \overline{f}^{(i)}_{l+\frac{1}{2},m} + \left(\frac{1+\sigma_l}{2}\right)\overline{\overline{f}}^{(i)}_{l,m} + \overline{f}^{(i)}_{l-\frac{1}{2},m}\right] + \overline{T}^{(i)}_{l,m}, \quad [l = 1(1)N, m = 1(1)M] \tag{21}$$

where $\overline{T}^{(i)}_{l,m} = O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)$.

We observe that for the uniform mesh case, i.e. when $\sigma_l = 1$, l=1(1)N, and $h_1 = h_2 = ... = h_{N+1} = h$ (say), the truncation error reduces to $\overline{T}_{l,m} = O(k^2h^2 + k^2h^4 + h^6)$ and thus, all the methods discussed above are of $O(k^2 + k^2h^2 + h^4)$.

4. CONVERGENCE OF THE ITERATIVE METHODS

Consider the following elliptic partial differential equation

$$u_{xx} + u_{yy} = \beta u_x, (x, y) \in \Omega \tag{22}$$

The above equation (22) is the steady state two-dimensional convection-diffusion equation, where $\beta = (1/\varepsilon) > 0$ is a constant, with ε (the perturbation parameter) being the ratio of convective velocity to the diffusion coefficient.

We apply the difference scheme (11) with $\overline{T}_{l,m} = 0$ to the above equation, considering the constant mesh case by taking $\sigma_l = 1$, for l =1(1)N and letting $p = k/h$ and $R = (\beta h/2) >$ 0, which is called the *cell Reynolds number*, to obtain

$$\begin{aligned}&(1+R)u_{l-1,m-1} + 10u_{l,m-1} + (1-R)u_{l+1,m-1} + (12p^2 - 2 + 12p^2R + 4p^2R^2 - 2R)u_{l-1,m}\\ &-(24p^2 + 20 + 8p^2R^2)u_{l,m} + (12p^2 - 2 - 12p^2R + 4p^2R^2 + 2R)u_{l+1,m}\\ &+(1+R)u_{l-1,m+1} + 10u_{l,m+1} + (1-R)u_{l+1,m+1} = 0, [l = 1(1)N, m = 1(1)M]\end{aligned} \tag{23}$$

The above is a system of *NM* number of linear equations in *NM* number of unknowns, which may be expressed in the matrix form as $\boldsymbol{Au = 0}$, where

$\boldsymbol{u} = \left[u_{1,1}, u_{2,1}, ..., u_{N,1}, u_{1,2}, u_{2,2}, ..., u_{N,2}, ..., u_{1,M}, u_{2,M}, ..., u_{N,M}\right]^{\mathrm{T}}$,

$\boldsymbol{A} = \left[\boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{P}\right]_{NM\times NM}$, (Tri-block-diagonal Matrix)

$\boldsymbol{P} = [1+R, 10, 1-R]_{N\times N}$, (Tri-diagonal Matrix)

$\boldsymbol{Q} = [12p^2 - 2 + 12p^2R + 4p^2R^2 - 2R, -(24p^2 + 20 + 8p^2R^2), 12p^2 - 2 - 12p^2R + 4p^2R^2 + 2R]_{N\times N}$
(Tri-diagonal Matrix)

Now, applying the Jacobi Iteration Method to the above system of equations, we obtain

$$\begin{aligned}&(24p^2 + 20 + 8p^2R^2)u_{l,m}^{(s+1)} =\\ &(1+R)u_{l-1,m-1}^{(s)} + 10u_{l,m-1}^{(s)} + (1-R)u_{l+1,m-1}^{(s)} + (12p^2 - 2 + 12p^2R + 4p^2R^2 - 2R)u_{l-1,m}^{(s)}\\ &+(12p^2 - 2 - 12p^2R + 4p^2R^2 + 2R)u_{l+1,m}^{(s)} + (1+R)u_{l-1,m+1}^{(s)} + 10u_{l,m+1}^{(s)} + (1-R)u_{l+1,m+1}^{(s)}\end{aligned} \tag{24}$$

where s = 0, 1, 2,…

We examine the stability of (24) by assuming that an error $\varepsilon_{l,m}^{(s)}$ exists at each grid point (x_l, y_m) at the sth iteration. The corresponding error equation at the sth iteration is given by

$$\begin{aligned}&(24p^2+20+8p^2R^2)\varepsilon_{l,m}^{(s+1)} = \\ &\quad (1+R)\varepsilon_{l-1,m-1}^{(s)}+10\varepsilon_{l,m-1}^{(s)}+(1-R)\varepsilon_{l+1,m-1}^{(s)}+(12p^2-2+12p^2R+4p^2R^2-2R)\varepsilon_{l-1,m}^{(s)} \\ &+(12p^2-2-12p^2R+4p^2R^2+2R)\varepsilon_{l+1,m}^{(s)}+(1+R)\varepsilon_{l-1,m+1}^{(s)}+10\varepsilon_{l,m+1}^{(s)}+(1-R)\varepsilon_{l+1,m+1}^{(s)}\end{aligned} \tag{25}$$

We analyze the behavior of the error $\varepsilon_{l,m}^{(s)}$ by assuming it to be of the form

$$\varepsilon_{l,m}^{(s)} = \xi^s A^l B^m \sin\left(\frac{\pi a l}{N+1}\right)\sin\left(\frac{\pi b m}{M+1}\right), \quad 1 \le a \le N,\ 1 \le b \le M \tag{26}$$

where A and B are arbitrary constants and ξ is the propagating factor which determines the rate of growth or decay of the errors. The necessary and sufficient condition for the iterative method to be stable is $|\xi| < 1$.

Using (26) in (25), the propagating factor for the Jacobi iteration method is obtained as

$$\begin{aligned}&\xi_J = \\ &\frac{\cos\left(\frac{\pi a}{N+1}\right)\left\{(1-R^2)\left[\cos\left(\frac{\pi b}{M+1}\right)+6p^2-1\right]^2+4p^4R^4+4p^2R^2\left[\cos\left(\frac{\pi b}{M+1}\right)+6p^2-1\right]\right\}}{\left\{\left[\cos\left(\frac{\pi b}{M+1}\right)+6p^2-1+2p^2R^2\right]^2-\left[R\cos\left(\frac{\pi b}{M+1}\right)+6p^2R-R\right]^2\right\}^{1/2}\left(6p^2+5+2p^2R^2\right)} \\ &\qquad +\frac{5\cos\left(\frac{\pi b}{M+1}\right)}{\left(6p^2+5+2p^2R^2\right)},\ 1 \le a \le N,\ 1 \le b \le M\end{aligned} \tag{27}$$

Thus, the Jacobi Iteration method is stable for those values of R such that $|\xi_J| < 1$.

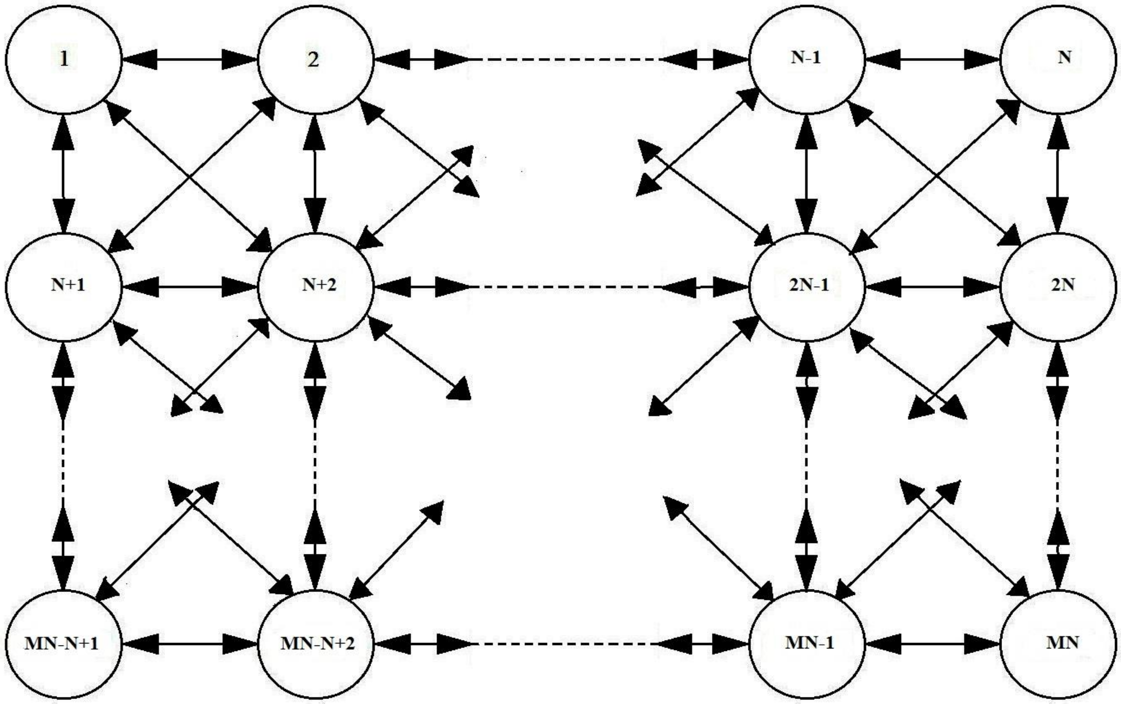


Fig.2. Directed Graph

Similarly, applying the Gauss-Siedal iteration method to (23) and assuming the error at each grid point (x_l, y_m) at the sth iteration to be of the form (26), the corresponding propagation factor ξ_{GS} is given by the equation

$$\eta^3 - \left[10\psi + (1-R)(6p^2 - 1 + 2p^2R^2 + 6p^2R - R)\cos\left(\tfrac{\pi b}{M+1}\right)\phi^2\right]\eta^2 +$$
$$\left[25\psi^2 - (1-R^2)\cos^2\left(\tfrac{\pi b}{M+1}\right)\phi^2 - (6p^2 - 1 + 2p^2R^2 + 6p^2R - R)(6p^2 - 1 + 2p^2R^2 - 6p^2R + R)\phi^2\right]\eta$$
$$-\phi^2(1+R)\left(6p^2 - 1 + 2p^2R^2 - 6p^2R + R\right)\cos\left(\tfrac{\pi b}{M+1}\right) = 0, 1 \le a \le N,\ 1 \le b \le M, \quad (28)$$

where $\eta = \xi_{GS}^{1/2}$, $\phi = \dfrac{\cos\left(\frac{\pi a}{N+1}\right)}{6p^2 + 5 + 2p^2R^2}$ and $\psi = \dfrac{\cos\left(\frac{\pi b}{M+1}\right)}{6p^2 + 5 + 2p^2R^2}$.

Thus, the Gauss-Siedal iteration method is stable for those values of R such that $|\xi_{GS}| < 1$.

Now, for the coefficient matrix $\boldsymbol{A}$ to be diagonally dominant, we must have

$$\left|24p^2 + 20 + 8p^2R^2\right| \ge |1+R| + 10 + |1-R| + \left|(12p^2 - 2)(1+R) + 4p^2R^2\right|$$
$$+\left|(12p^2 - 2)(1-R) + 4p^2R^2\right| + |1+R| + 10 + |1-R| \quad (29)$$

Assuming the diffusion dominated case i.e. $R \le 1$ and taking $p \ge 1/\sqrt{6}$, it is easy to see that relation (29) is satisfied. Also, clearly, strict inequality holds for the first row of the matrix ***A***. Hence ***A*** is diagonally dominant. Also, the Directed Graph of ***A*** is strongly connected (see Fig. 2). Hence the matrix ***A*** is irreducibly diagonally dominant for $R \le 1$ and $p \ge 1/\sqrt{6}$ (Varga, 2000). Thus, under these conditions, Jacobi and Gauss-Siedal Iteration methods are convergent for any initial guess.

5. COMPUTATIONAL IMPLEMENTATION

We divided the interval [0, 1] in y-direction into (M+1) parts of equal lengths $k > 0$, so that $y_m = mk$ for m = 0(1)M+1. Further, the interval [0, 1] in x-direction is divided into (N+1) parts with $0 = x_0 < x_1 < x_2 < \ldots < x_{N+1} = 1$, where $h_{l+1} = x_{l+1} - x_l$, for l = 0(1)N and $\sigma_l = (h_{l+1}/h_l) > 0$, for l = 1(1)N. This discretizes the solution domain Ω with grid points given by (x_l, y_m), l = 0(1)N+1, m = 0(1)M+1.

Now,
$$\begin{aligned} 1 &= x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + \ldots + (x_1 - x_0) \\ &= h_{N+1} + h_N + \ldots + h_1 \\ &= (1 + \sigma_1 + \sigma_1\sigma_2 + \ldots + \sigma_1\sigma_2 \ldots \sigma_N) h_1, \end{aligned} \tag{30}$$

which gives the value of the first step length in x-direction as :

$$h_1 = 1/(1 + \sigma_1 + \sigma_1\sigma_2 + \ldots + \sigma_1\sigma_2 \ldots \sigma_N) \tag{31}$$

Using the above value, we are able to determine the values of subsequent step-lengths as $h_{l+1} = \sigma_l h_l$, l = 1(1)N. Hence we determine each grid point (x_l, y_m) of the rectangular mesh.

For the sake of simplicity, we assume here that $\sigma_l = \sigma$ (constant) for all l = 1(1)N, so that

$$h_1 = (1 - \sigma)/(1 - \sigma^{N+1}) \tag{32}$$

Thus having prescribed the total number of mesh points in the x- direction, say, N+2, we can determine the first step length on the left using (32), and further step lengths are determined by using the relation $h_{l+1} = \sigma h_l$, $l = 1(1)N$. For uniform mesh case, i.e., for $h_{l+1} = h_l$ = h, l = 1(1)N, we obtain the corresponding $O(k^2 + k^2h^2 + h^4)$ finite difference scheme.

Substituting the approximations (20.2), (20.5), (20.8) and (20.9) in the given system of differential equations (1), we obtain a variable mesh method of $O(k^2 + h_l)$ as

$$\overline{U}^{(i)}_{xxl,m} + \overline{U}^{(i)}_{yyl,m} = f^{(i)}(x_l, y_m, U^{(1)}_{l,m}, U^{(2)}_{l,m}, ..., U^{(n)}_{l,m}, \overline{U}^{(1)}_{xl,m}, \overline{U}^{(2)}_{xl,m}, ..., \overline{U}^{(n)}_{xl,m}, \overline{U}^{(1)}_{yl,m}, \overline{U}^{(2)}_{yl,m}, ..., \overline{U}^{(n)}_{yl,m})$$
$$+ O(k^2 + h_l); \qquad [l = 1(1)N,\ m = 1(1)M] \tag{33}$$

Note that, for constant mesh case, i.e., for $\sigma_l = 1$, the method (33) becomes a constant mesh method of $O(k^2 + h^2)$. In this section, we have solved two linear and two non-linear problems to which the exact solutions have been prescribed. The right hand side functions and the boundary conditions are determined using the exact solutions. We have compared the numerical results of the proposed method (21) with the corresponding numerical results obtained by using the method (33). The linear difference equations are solved by the Gauss-Siedel method and the non-linear ones by the Newton-Raphson method. The iterations were terminated once the absolute error tolerance $\le 10^{-12}$ was achieved. All the computations are carried out using MATLAB programming language.

Problem 1: To solve the convection-diffusion equation (22) whose exact solution is given by

$u(x,y) = e^{\frac{\beta x}{2}} \dfrac{\sin \pi y}{\sinh \gamma}\left[2e^{-\frac{\beta}{2}} \sinh \gamma x + \sinh \gamma (1-x)\right]$, where $\gamma^2 = \pi^2 + \dfrac{\beta^2}{4}$. The maximum absolute errors (MAE) of u for $\sigma = 0.92$ and $\sigma = 1$ are tabulated in Table 1a and 1b respectively. Figure 3 gives a comparison of the plots of the exact and numerical solutions for $\beta = 1000$.

Problem 2: (*Poisson's equation in polar coordinates*)

$$u_{rr} + \frac{\alpha}{r} u_r + u_{zz} = G(r,z), \quad 0 < r, z < 1 \tag{34}$$

For $\alpha = 1$, the above equation represents the two-dimensional Poisson's equation in cylindrical polar coordinates in *r*-*z* plane. The exact solution is given by $u = \cosh r \cosh z$. The MAE of u with $\sigma = 1.4$ are tabulated in Table 2a for $\alpha = 1$ and 2. Table 2b gives the MAE of u with $\sigma = 1$ for fixed value of mesh ratio parameter $\lambda = k / h^2$ =20. Figure 4 gives the plots of the exact and numerical solutions of Problem 2.

Problem 3: (*Non-linear Convection Equation*)

$$\varepsilon(u_{xx} + u_{yy}) = u(u_x + u_y) + g(x,y), 0 < x, y < 1 \tag{35}$$

The exact solution is given by $u = e^x \sin\left(\frac{\pi y}{2}\right)$. The MAE of u for variable and constant mesh cases are tabulated in Table 3a & 3b respectively. Figure 5 gives a comparison of the plots of the exact and numerical solutions of Problem 3.

Problem 4: (2D *steady-state Navier Stokes' model equations in Cartesian coordinates*)

$$\tfrac{1}{R_e}(u_{xx}+u_{yy}) = uu_x + vu_y + f(x,y),\ 0<x,y<1 \tag{36a}$$

$$\tfrac{1}{R_e}(v_{xx}+v_{yy}) = uv_x + vv_y + g(x,y),\ \ 0<x,y<1 \tag{36b}$$

$$0 = u_x + v_y \tag{36c}$$

where $R_e > 0$ is a constant and is called the Reynolds number. The exact solutions are $u = \sin(\pi x)\sin(\pi y)$, $v = \cos(\pi x)\cos(\pi y)$. The MAEs of u and v are tabulated in Table 4a for $\sigma = 0.92$ and in Table 4b for $\sigma = 1$ and fixed value of mesh ratio parameter λ = 20. Figure 6 gives a comparison of the plots of the exact and numerical solutions.

Table 1a: The MAE ($\sigma = 0.92$) – variable mesh case

	Proposed $O(k^2 + k^2 h_l + h_l^3)$ methods			$O(k^2 + h_l)$ method		
(N,M)	β =100	β =1000	β = 1400	β =100	β =1000	β = 1400
(30,30)	6.1173(-04)	2.6871(-01)	4.1026(-01)	9.1907(-01)	9.7583(-01)	9.7351(-01)
(40,40)	9.2476(-05)	4.5122(-02)	1.0026(-01)	8.5900(-01)	9.5874(-01)	9.7085(-01)
(50,50)	3.9142(-05)	3.3946(-03)	1.0118(-02)	8.0263(-01)	9.3978(-01)	9.4924(-01)
(60,60)	2.4982(-05)	2.3368(-04)	5.9052(-04)	7.5800(-01)	9.0396(-01)	9.2049(-01)
(70,70)	1.8386(-05)	3.3685(-05)	6.5273(-05)	7.2960(-01)	8.6352(-01)	8.8457(-01)
(80,80)	1.4568(-05)	1.1038(-05)	1.5365(-05)	7.1495(-01)	8.2419(-01)	8.4427(-01)

Table 1b: The MAE ($\sigma = 1$) – constant mesh case

	Proposed $O(k^2 + k^2 h^2 + h^4)$ methods			$O(k^2 + h^2)$ method		
h	β =100	β =1000	β = 1400	β =100	β =1000	β = 1400
$\frac{1}{30}$	4.0098(-02)	7.0453(-01)	7.7859(-01)	Oscillations	Oscillations	Oscillations
$\frac{1}{40}$	1.7223(-02)	6.2489(-01)	7.1473(-01)	Oscillations	Oscillations	Oscillations
$\frac{1}{50}$	8.2806(-03)	5.5427(-01)	6.5603(-01)	Oscillations	Oscillations	Oscillations
$\frac{1}{60}$	4.3251(-03)	4.9167(-01)	6.0216(-01)	Oscillations	Oscillations	Oscillations
$\frac{1}{70}$	2.4556(-03)	4.3622(-01)	5.5273(-01)	Oscillations	Oscillations	Oscillations
$\frac{1}{80}$	1.4675(-03)	3.8712(-01)	5.0738(-01)	Oscillations	Oscillations	Oscillations

A comparison of the numerical results in Tables 1a & 1b indicates that a variable mesh produces significantly better results for large values of β than the corresponding uniform mesh case. On the other hand, although the lower order variable mesh method generates oscillation free results, the corresponding uniform mesh method fails totally.

Table 2a: The MAE ($\sigma = 1.4$) - variable mesh case

	Proposed $O(k^2 + k^2 h_l + h_l^3)$ methods		$O(k^2 + h_l)$ method	
(N, M)	α=1	α=2	α=1	α=2
(30,30)	3.4369(-05)	5.1954(-05)	7.6763(-02)	1.2481(-01)
(40,40)	3.0933(-05)	4.7884(-05)	7.6952(-02)	1.2578(-01)
(50,50)	2.9326(-05)	4.6155(-05)	7.6990(-02)	1.2610(-01)
(60,60)	2.8463(-05)	4.5383(-05)	7.6998(-02)	1.2620(-01)
(70,70)	2.7938(-05)	4.4922(-05)	7.7007(-02)	1.2624(-01)
(80,80)	2.7600(-05)	4.4617(-05)	7.7004(-02)	1.2625(-01)

Table 2b: The MAE ($\sigma = 1$, $\lambda = 20$) – constant mesh case

	Proposed $O(k^2 + k^2 h^2 + h^4)$ methods		$O(k^2 + h^2)$ method	
h	α=1	α=2	α=1	α=2
$\frac{1}{10}$	3.5818(-04)	3.7441(-04)	Oscillations	Oscillations
$\frac{1}{20}$	2.3422(-05)	2.4239(-05)	Oscillations	Oscillations
$\frac{1}{40}$	1.4823(-06)	1.5200(-06)	Oscillations	Oscillations

Table 3a: The MAE ($\sigma = 0.92$) – variable mesh case

	Proposed $O(k^2 + k^2 h_l + h_l^3)$ methods		$O(k^2 + h_l)$ method	
(N, M)	ε =0.1	ε =0.01	ε =0.1	ε =0.01
(30,30)	2.7145(-04)	8.0162(-04)	1.3032(-01)	2.4512(-01)
(40,40)	1.5411(-04)	4.9423(-04)	1.2397(-01)	2.2878(-01)
(50,50)	1.0010(-04)	3.7439(-04)	1.2138(-01)	2.1490(-01)
(60,60)	7.0922(-05)	3.3317(-04)	1.2020(-01)	2.0296(-01)
(70,70)	5.3450(-05)	3.1724(-04)	1.1957(-01)	1.9230(-01)
(80,80)	4.2179(-05)	3.0890(-04)	1.1938(-01)	1.8249(-01)

Table 3b: The MAE ($\sigma = 1,\ \lambda = 20$)– constant mesh case

	Proposed $O(k^2 + k^2h^2 + h^4)$ methods		$O(k^2 + h^2)$ method	
h	ε =0.1	ε =0.01	ε =0.1	ε =0.01
$\frac{1}{10}$	1.0281(-02)	9.7481(-02)	Oscillations	Oscillations
$\frac{1}{20}$	6.1051(-04)	1.1025(-03)	Oscillations	Oscillations
$\frac{1}{40}$	3.8190(-05)	6.2725(-05)	Oscillations	Oscillations

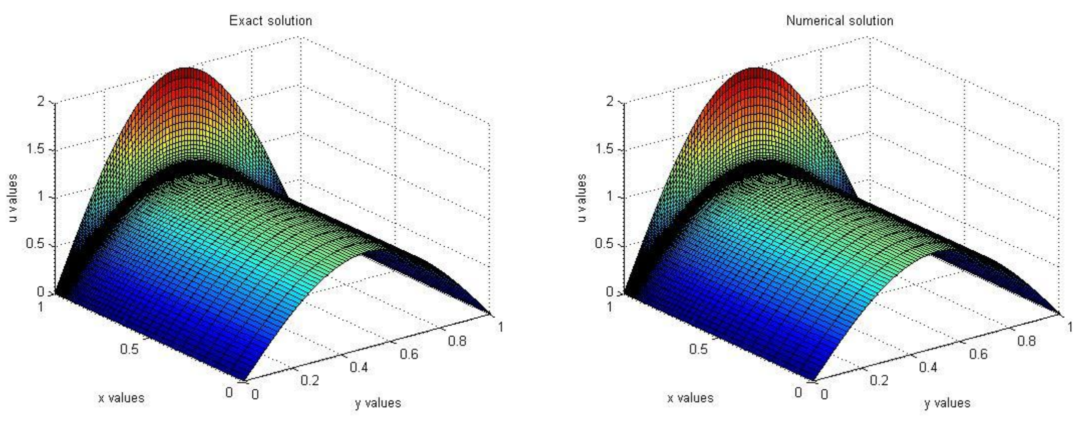
Table 4a: The MAE ($\sigma = 0.92$)– variable mesh case

		Proposed $O(k^2 + k^2h_l + h_l^3)$ methods			$O(k^2 + h_l)$ method		
(N, M)		R_e =10	R_e =50	R_e =100	R_e =10	R_e =50	R_e =100
(50,50)	u	2.2913(-04)	8.0941(-04)	3.4184(-03)	5.6489(-01)	Oscillations	Oscillations
	v	1.4789(-04)	1.8284(-03)	4.9054(-03)	1.7010(-01)	Oscillations	Oscillations
(60,60)	u	1.6704(-04)	7.7948(-04)	3.1933(-03)	5.6227(-01)	Oscillations	Oscillations
	v	1.2777(-04)	1.6766(-03)	4.5701(-03)	1.6940(-01)	Oscillations	Oscillations
(70,70)	u	1.2995(-04)	7.6902(-04)	3.1160(-03)	5.6106(-01)	Oscillations	Oscillations
	v	1.1651(-04)	1.5934(-03)	4.3827(-03)	1.6903(-01)	Oscillations	Oscillations
(80,80)	u	1.0613(-04)	7.6602(-04)	3.0840(-03)	5.6052(-01)	Oscillations	Oscillations
	v	1.0952(-04)	1.5437(-03)	4.2679(-03)	1.6884(-01)	Oscillations	Oscillations

Table 4b: The MAE ($\sigma = 1,\ \lambda = 20$)– constant mesh case

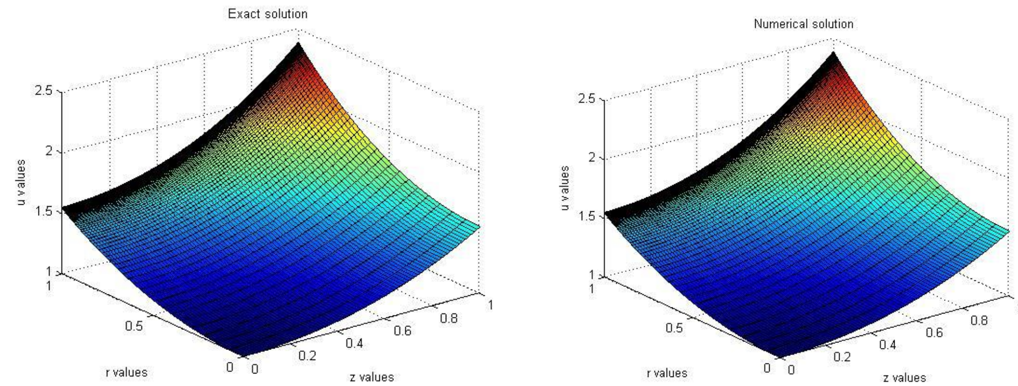
		Proposed $O(k^2 + k^2h^2 + h^4)$ methods			$O(k^2 + h^2)$ method		
h		R_e =10	R_e =50	R_e =100	R_e =10	R_e =50	R_e =100
$\frac{1}{20}$	u	1.2387(-03)	3.2983(-03)	5.0097(-03)	Oscillations	Oscillations	Oscillations
	v	4.8642(-04)	2.6024(-03)	5.2111(-03)	Oscillations	Oscillations	Oscillations
$\frac{1}{40}$	u	7.7636(-05)	2.0260(-04)	3.0227(-04)	Oscillations	Oscillations	Oscillations
	v	3.0524(-05)	1.5937(-04)	3.1592(-04)	Oscillations	Oscillations	Oscillations

For the fixed value of mesh ratio parameter $\lambda = k/h^2$, i.e., $k = \lambda h^2$, the uniform mesh $O(k^2 + k^2h^2 + h^4)$ method becomes fourth order accurate in space. This order of accuracy can be verified from Tables 2b, 3b and 4b, using the formula $\frac{\log(E_{h_1}/E_{h_2})}{\log(h_1/h_2)}$, where E_{h_1} and E_{h_2} are the MAEs for two uniform mesh widths h_1 and h_2, respectively. For example, in Table 3b, let us choose h_1=1/20, h_2=1/40, ε=0.1 with the corresponding MAEs 6.1051E-04 and 3.8190E-05. Using above formula it is easy to verify that the order of accuracy of the proposed method indeed is $3.99 \cong 4.0$. Note that, the above formula can be used only in constant mesh case and cannot be used to calculate the order of accuracy in variable mesh case.



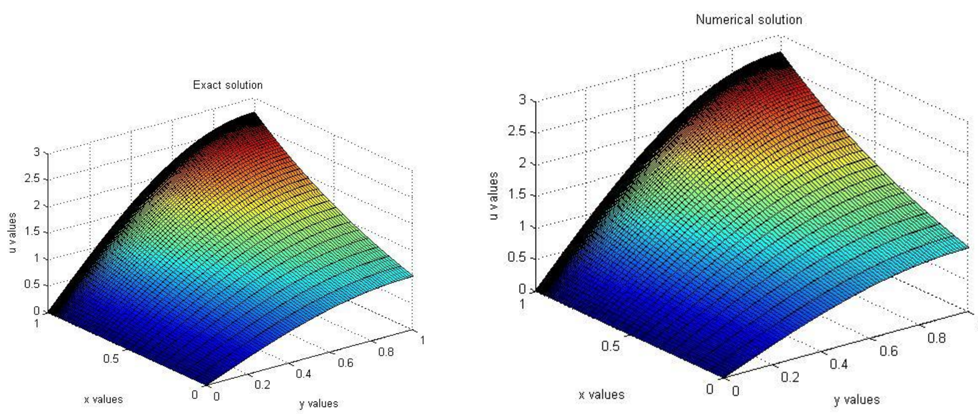
Exact Solution Numerical Solution

Figure 3: Exact and Numerical Solution of *Convection-Diffusion Equation* (22) at $\beta = 1000$.



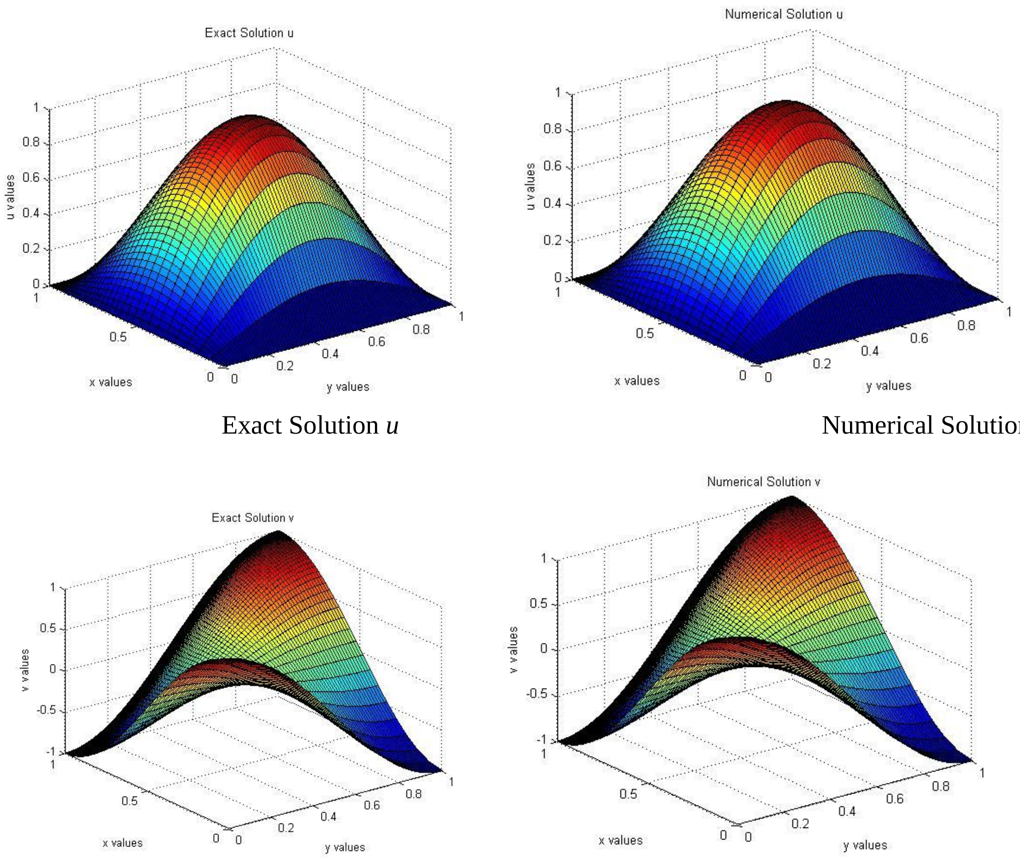
Exact Solution Numerical Solution

Figure 4: Exact and Numerical Solution of *Poisson's Equation* (34) in cylindrical Polar coordinates in *r*-*z* plane.



Exact Solution Numerical Solution

Figure 5: Exact and Numerical Solution of *Non-linear convection-Diffusion Equation* (35) at $\varepsilon = 0.01$.



Exact Solution *u* Numerical Solution *u*

Exact Solution *v* Numerical Solution *v*

Figure 6: Exact and Numerical Solutions of *Navier Stokes Equations in Cartesian Coordinates* (36) at $R_e = 100$.

6. CONCLUDING REMARKS

In this paper, we have developed a high order finite difference method based on off-step discretization for the system of 2D non-linear elliptic boundary value problems on a variable mesh, using 9 grid points and 3 function evaluations. This new variable mesh strategy results in solving the tri-block-diagonal system of difference equations. Numerical experiments have been made to compare the proposed variable mesh methods with the variable mesh method of $O(k^2 + h_l)$. We have solved four benchmark problems of physical significance. We observed that unlike the case of high order constant mesh techniques, our methods work successfully for the small values of the perturbation parameter ε for the solution of the steady state convection diffusion equation. The numerical results show that the proposed methods do not produce any numerical oscillations when applied to the Navier Stokes' model equations for high values of Reynolds number, whereas the corresponding lower order variable mesh method is unstable for large Reynolds number.

References

1. Yavneh, I.V. (1997). Analysis of a fourth-order compact scheme for convection diffusion. J. Comput. Phys., v. 133, pp. 361-364.

2. Zhang, J. (1997). On Convergence of iterative methods for a fourth-order discretization scheme. Appl. Math. Lett., v. 10, pp. 49-55.

3. Zhang, J. (1998). On convergence and performance of iterative methods with fourth order compact scheme. Numer. Meth. Partial Diff. Eq., v. 14, pp. 263-280.

4. Spotz, W.F., & Carey, G.F. (1995). High-order compact scheme for the steady-stream function vorticity equations. Int. J. Numer. Meth. Engg., v. 38, pp. 3497-3512.

5. Sakurai, K., Aoki, T., Lee, W.H., & Kato, K. (2002). Poisson equation solver with fourth-order accuracy by using interpolated differential operator scheme. Comput. Math. Applics., vl. 43, pp. 621-630.

6. Jain, M.K., Jain, R.K., & Krishna, M. (1994). Fourth order difference method for quasi-linear poisson equation in cylindrical symmetry. Commn. Numer. Meth. Engg., v. 10, pp. 291-296.

7. Mohanty, R.K., Karaa, S., & Arora, U. (2006). Fourth order nine point unequal mesh discretization for the solution of 2D non-linear elliptic partial differential equations. Neural Parallel & Scientific Computations, v. 14, pp. 453-470.

8. Chawla, M.M., & Shivakumar, P.N. (1996). An efficient finite difference method for two-point boundary value problems. Neural Parallel & Scientific Computations, v. 4, pp. 387-396.

9. Jain, M.K., Iyengar, S.R.K., & Subramanyam, G.S. (1983). Variable mesh methods for the numerical solution of two-point singular perturbation problems. Comp. Meth. Appl. Mech. Engg., v. 42, pp. 273-286.

10. Mohanty, R.K. (2005). A family of variable mesh methods for the estimates of $(\partial u / \partial r)$ and solution of non-linear two point boundary value problems with singularity. J. Comp. Appl. Math., v. 182, pp. 173-187.

11. Kadalbajoo, M.K., & Kumar, D. (2010). Variable mesh finite difference method for self-adjoint singularly perturbed two-point boundary value problems. J. Comp. Math., v. 28, 5, pp. 711-724.

12. Jain, M.K., Jain, R.K., & Mohanty, R.K. (1989). A fourth order difference method for elliptic equations with non-linear first derivative terms. Numer. Meth. Partial Diff. Eq., v. 5, pp. 87-95.

13. Jain, M.K., Jain, R.K., & Mohanty, R.K. (1991). Fourth order difference methods for the system of 2D nonlinear elliptic partial differential equations. Numer. Meth. Partial Diff. Eq., v. 7, pp. 227-244.

14. Mohanty, R.K. (1997). Order h^4 difference methods for a class of singular two space elliptic boundary value problems. J. Comp. Appl. Math., v. 81, pp. 229-247.

15. Mohanty, R.K., & Singh, S. (2006). A new fourth order discretization for singularly perturbed two dimensional non linear elliptic boundary value problems. Appl. Math. Comput., v. 175, pp. 1400-1414.

16. Saldanha, G. (2001). Technical Note: A fourth order finite difference scheme for a system of 2D nonlinear elliptic partial differential equations. Numer. Meth. Partial Diff. Eq., v. 17, pp. 43-53.

17. Ananthakrishnaiah, U., & Saldanha, G. (1995). A fourth-order finite difference scheme for two dimensional nonlinear elliptic partial differential equations. Numer. Meth. Partial Diff. Eq., v. 11, pp. 33-40.

18. Mohanty, R.K., & Setia, N. (2012). A new fourth order compact off-step discretization for the system of 2D nonlinear elliptic partial differential equations. East Asian J. Appl. Math., v. 02, pp. 59-82.

19. Kelly, C.T. (1995). Iterative Methods for Linear and Non-Linear Equations. SIAM Publications, Philadelphia.

20. Varga, R.S. (2000). Matrix Iterative Analysis. Springer, New York.

21. Saad, Y. (2003). Iterative Methods for Sparse Linear Systems. SIAM Publications, Philadelphia.

22. Hageman, L.A., & Young, D.M. (2004). Applied Iterative Methods. Dover Publication, New York.